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DYNAMIC CONSISTENCY IN ENGLISH AUCTIONS AND EXPECTED UTILITY THEORY

by

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1. Introduction

This paper explores the relationship between dynamically consistent behavior and expected utility maximizing behavior in the context of English auctions.[1] We consider English auctions where: (a) the objects offered for sale are lotteries, that is, risky prospects with objectively known probabilities and (b) the conditions of the independent private values model prevail, that is, each participant in the auction knows his own assessment of the value of the object that is being auctioned but not its value to other bidders, the individual evluations of the object are independent, and the behavior of the bidders is non-cooperative.

The main result of this paper is that decision makers are dynamically consistent English auction bidders if and only if they are expected utility maximizers. The proof of this result involves two intermediary steps that are of interest in their own right: (a) dynamic consistency implies the existence of a dominant bidding strategy, and (b) this strategy is preference revealing in the sense that the actual bid, namely the price at which a dynamically consistent decision maker withdraws from the auction, coincides with his personal assessment of the value of the lottery being auctioned.

Johnsen and Donaldson [1985] examined the structure of preference relations that are implied by dynamic consistency. They show that time consistent planning by itself does not imply expected utility maximizing behavior; additional structure is necessary. In the present analysis, this structure is imposed by the rules of the auction and by the nature of the preference relations under consideration.

Weller [1978], using general decision trees and assuming that at each node of the decision tree the decision maker's preferences on the remaining uncertainty are representable by a linear functional, shows that dynamically consistent behavior is equivalent to expected utility maximizing behavior. Our result is stronger than that of Weller in two respects: it does not require that preferences be linear in the probabilities and it is obtained with the use of proper subsets of the set of all decision trees.

Hammond [1985], using the set of all finite decision trees as a universe of discourse, shows that dynamic consistency and the existence of a choice function from finite sets of consequences (which in his case is the set of simple probability distributions over a fixed set of outcomes) into choice subsets imply that preferences on the space of probability distributions are linear. Instead of choice function our analysis is based on the existence of a "smooth" preference ordering on the set of all lotteries on $\mathbb{R}$, the set of real numbers. More importantly, our results require only that decision makers be dynamically consistent over decision trees that are induced by the structure of English auctions.

Epstein [1985] studied the aforementioned issue within the framework of infinite horizon temporal lotteries. His analysis shows that dynamic consistency and indifference to the timing of resolution of uncertainty imply expected utility maximizing behavior. As Epstein's substantive use of real time does not have a counterpart in our model, the two theories are complementary.

2. The Model

Let L be the set of all cumulative probablitiy distribution functions over $\mathbb{R}$ endowed with the topology of weak convergence and metrized by the Levy metric.[2] We denote by Ω be the set of all weak order relations on L that are monotone (in the sense of first order stochastic dominance), continuous, and "smooth" in the sense of having Hadamard differentiable representation $V:L \to \mathbb{R}$.[3]

For each $r > 0$ and continuous $F \in L$ with Supp $F = [0,r]$ let F^y and f^y, $y > 0$, be the cumulative distribution function with Supp $F^y = [y, r]$ and the respective density function obtained from F by the application of Bayes' rule. Let

(1) $$\Delta^r = \{(y, x) \in [0, r]^2 \mid y \leq x\}$$

and define $G:\Delta^r \times L \times \mathbb{R} \times L \longrightarrow L$ by

(2) $$G(y,x,A,w,F) = [1 - F^y(x)]\delta_w + \int_y^x (A - z)f^y(z)dz,$$

where δ_w is the distribution in L that assigns the entire probability mass to $w \in \mathbb{R}$ and $(A - z)$ is the distribution in L defined by $(A - z)(x) = A(x + z)$.

Definition 1: A decision problem for a decision maker with preference relation in Ω is a triplet (A, w, F) in $L \times \mathbb{R} \times L$ where F is continuous and Supp $F = [0, r]$, $r > 0$, A dominates δ_w and δ_w dominates $(A - r)$ according to first order stochastic dominance.

The distribution A is called the initial distribution and w is referred to as the level of wealth. In the context of English auctions the decision problem has the following interpretation: the level of wealth, w, is the decision maker's initial wealth, (A - w) is the object being auctioned, and F is the decision maker's prior probability distribution over the maximal bid of the other participants.

For each $(y, x) \in \Delta^r$, $G^{y,x} = G(y, x, A, w, F)$ represents the payoff to a decision maker who, when the price y is announced, plans to quit at x. To see this recall that according to the rules of the auction such a participant receives δ_w in the event that at least one other participant stays in the auction when x is announced, i.e., he receives δ_w with probability $[1 - F^y(x)]$, and he receives (A - z) if the highest bidder among the other participants quits when $z \leq x$ is announced, i.e., he receives (A - z) with density $f^y(z)$.

Consider a decision maker with preference relation in Ω. Denote by V the functional representation of his preferences, and let (A, w, F) be his decision problem.

Assumption: The decision to quit at $y \geq 0$ is made according to the rule

(3) quit at y iff $V(\delta_w) \geq V(G^{y,x})$ for all $x \geq y$.

Note that $\delta_w = G^{y,y}$ and that the decision maker stays in the auction at y as long as there exists at least one x such that $V(G^{y,x}) > V(\delta_w)$. It can be shown, using first order stochastic dominance, that a decision maker never finds it optimal to quit at one stage and reenter the bidding at a latter stage. Thus, without loss of generality, we assume that the decision to quit is irreversible.[4]

Definition 2: A decision maker is said to be dynamically consistent if for each decision problem (A,w,F) there exists b in [0, r] such that

(4)

(a) $V(G^{y,b}) \geq V(G^{y,x})$ for all $y \in [0, b]$, and $x \in [y, r]$

(b) $V(G^{y,b}) > V(G^{y,y})$ for all $y \in [0, b)$.

In other words a decision maker is dynamically consistent if for every decision problem there exists a unique number b such that whenever y < b quitting at b is optimal and strictly better than quitting immediately. The definition implies that a dynamically consistent decision maker quits the auction when the announced price reaches the value b and that this is what he is planning on doing all along. For such a decision maker we call the price b the actual bid and the corresponding distribution, (A − b), the stopping distribution. Finally, the value of A at w is the unique v that satisfies

(5) $$V(A - v) = V(\delta_w).$$

3. Results

To begin with we show that dynamic consistency implies that the stopping distribution is independent of the distribution F in the decision problem. In other words, a dynamically consistent decision maker with a preference relation in Ω has a dominant English auction bidding strategy.

PROPOSITION 1: *Assume a dynamically consistent decision maker with a preference relation in Ω. Then all decision problems with the same initial distribution and the same level of wealth have the same stopping distribution, B, independently of F.*

Proof: Let V represent the decision maker's preferences, let (A, w, F) be a decision problem, and let b be the corresponding actual bid. For every $y \in [0, b]$ let $\partial_x G^{y,b}$ denote the tangent to the path $\{G^{y,x} \mid x \geq y\}$ at x = b and let $\partial_y G^{b,b}$ be the (inward) tangent to the path $\{G^{y,b} \mid y \leq b\}$ at $y = b$. These tangents are given by

$$\partial_x G^{y,b} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} [G^{y,x+\epsilon} - G^{y,x}] = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \Big[[1 - F^y(x+\epsilon)]\delta_w + \int_y^{x+\epsilon} (A - z) f^y(z)dz - [1 - F^y(x)]\delta_w - \int_y^x (A - z) f^y(z)dz \Big]$$

(6)

$$\partial_y G^{b,b} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} [G^{b-\epsilon,b} - G^{b,b}] = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \Big[[1 - F^{b-\epsilon}(b)]\delta_w + \int_{b-\epsilon}^b (A - z) f^{b-\epsilon}(z)dz - \delta_w \Big]$$

but $F^{b-\epsilon}(b) = \dfrac{F(b) - F(b - \epsilon)}{1 - F(b - \epsilon)}$, hence

$$\partial_x G^{y,b} = f^y(b)[(A - b) - \delta_w].$$

(7)

$$\partial_y G^{b,b} = f^b(b)[(A - b) - \delta_w]$$

Let u_w: $\mathbb{R} \to \mathbb{R}$ be the local utility function at δ_w. Since u_w is monotonic increasing in its first argument and since A dominates δ_w and δ_w dominates $(A - r)$ according to first order stochastic dominance, $\int u_w d(A - \delta_w) > 0$ and $\int u_w d((A - r) - \delta_w) < 0$. Let B be the unique distribution of the form $(A - x)$ for which $\int u_w d(B - \delta_w) = 0$. We need to show that $B = (A - b)$. This will conclude the proof since B is independent of F.

Since b is the actual bid, by dynamic consistency,

(8) $$0 \geq \frac{d}{dx} V(G^{b,x})\big|_b = \int u_w d(\partial_x G^{b,b}) = f^b(b) \int u_w d[(A - b) - \delta_w].$$

Since for all $y \in [0, b]$ $V(G^{y,b}) - V(\delta_w) \geq 0$ we have,

(9) $$0 \leq \lim_{\epsilon \to 0} \frac{1}{\epsilon} [V(G^{b-\epsilon,b}) - V(G^{b,b})] = \int u_w d(\partial_y G^{b,b}) = f^b(b) \int u_w d[(A - b) - \delta_w].$$

By (8) and(9), $\int u_w d[(A - b) - \delta_w] = 0$, and the stopping distribution $B = (A - b)$ is independent of F. //

Remark: By dynamic consistency, for all $y \in [0, b)$

(10) $$0 = \frac{d}{dx}V(G^{y,x})|_b = f^y(b)\int u_{y,b}d[(A - b) - \delta_w],$$

where $u_{y,b}$ is the local utility at $G^{y,b}$. Equation (10) will be used in the sequel.

Using Proposition 1, we show that dynamically consistent decision makers with preference relations in Ω always bid their values, i.e., their dominant bidding strategies are preference revealing.

PROPOSITION 2: *If a decision maker is dynamically consistent with preference relation in Ω, then for every decision problem (A, w, F), the stopping distribution, B, satisfies $V(B) = V(\delta_w)$.*

Proof: Let A be the initial distribution, let w be the level of wealth, and let v denote the value of A at w. Define $B = A - v$. Suppose that the actual bid, b, is smaller than v. Then by stochastic dominance $V(A - b) > V(B)$. By Proposition 1, b is independent of F. By continuity there exist $x \in (b, v)$ and a distribution F^b such that $V(G^{b,x}) > V(B)$. But $V(B) = V(\delta_w) = V(G^{b,b})$. Hence, $V(G^{b,x}) > V(G^{b,b})$. But this contradicts dynamic consistency. Thus, $b \geq v$.

Suppose that $b > v$. Then, by stochastic dominance, $V(A - b) < V(B)$. Take $y \in (v, b)$ and let

(11) $$\bar{\lambda} \equiv \max \{\lambda \in [0, 1] \mid V((1 - \lambda)\delta_w + \lambda(A - y)) \geq V(\delta_w)\}.$$

By stochastic dominance $V(A - y) < V(\delta_w)$ and hence $\bar{\lambda} < 1$. Choose F^y such that $F^y(b) > \bar{\lambda}$. Since $G^{y,b}$ is stocastichally dominated by

(12) $$Q \equiv (1 - F^y(b))\delta_w + F^y(b)(A - y)$$

we conclude that

(13) $$V(G^{y,b}) < V(Q) < V(\delta_w).$$

This contradicts dynamic consistency and thus b must be equal to v. But this implies that the stopping distribution is B, and B satisfies $V(B) = V(\delta_w)$. //

We turn now to the main result.

THEOREM: *A preference relation in Ω is dynamically consistent if and only if it has a linear representation $V: L \to \mathbb{R}$.*

Proof: The sufficiency part is immediate so its proof is omitted. To prove necessity we show that if V represents a dynamically consistent preference relation in Ω then its indifference sets in L are parallel, and thus V can be chosen to be linear.

Let w be in $\mathbb{R}$ and take B in L such that $V(B) = V(\delta_w)$. Consider a decision problem where the initial distribution is $A = B + m$, $m > 0$. By Proposition 2 the stopping distribution is B and, by Proposition 1, B must satisfy $\int u_w d(B - \delta_w) = 0$, where u_w is the local utility function at δ_w. Hence for every w

(14) $$V(B) = V(\delta_w) \Rightarrow \int u_w dB = u_w(w).$$

By continuity and stochastic dominance all the indifference sets of V are of the form

(15) $$I(\delta_w) = \{B \in L \mid \int u_w dB = u_w(w)\}.$$

To complete the proof we show that all the u_w's can be chosen to be equal to $u = u_0$. Take $w > 0$ (the case of $w < 0$ can be treated similarly) and let B be in $I(\delta_0)$. Choose $F \in L$ and $m > 0$ large enough so that $(B + m, 0, F)$ is a decision problem and $V(B + m) > V(\delta_w)$. This implies that for some $x > 0$ $V(G^{0,x}) > V(\delta_w)$. By Proposition 2 m is the actual bid. Hence, $V(G^{0,m}) = \max_{x \geq 0} V(G^{0,x}) > V(\delta_w)$. Also, $V(G^{m,m}) = V(\delta_0) < V(\delta_w)$. V is continuous along the path $\{G^{y,m} \mid y \in [0, m]\}$, and thus, there exist y such that $V(G^{y,m}) = V(\delta_w)$. But, by (10), this implies that $\int u_w d(B - \delta_0) = 0$. Hence, for all $B \in L$,

$$(16) \qquad \int u d(B - \delta_0) = 0 \text{ iff } \int u_W d(B - \delta_0) = 0,$$

where stochastic dominance is used for the "if" part of (16). Thus, u and u_W can be chosen to be equal. This implies that for all $A \in L$ $V(A) = \int u dA$. //

4. Discussion

In view of the restrictive nature of dynamic consistency it is important to recognize that *the negation of dynamic consistency is tantamount to self-deception*. Consider a decision maker whose preferences display dynamic inconsistency. In the context of English auctions described above such an individual decides whether to stay in the auction or to quit by comparing the utility of his wealth with the utility of an alternative bidding plan that he knows in advance he will not follow.[5] This self-deceiving attitude contradicts our intuitive notion of rational behavior. We note, however, that it is possible to maintain the assumption of dynamic consistency without necessarily foregoing the nonlinearity of the preference relations provided we forego the assumption of fixed preferences. In other words we can impose dynamic consistency by restricting consideration to bidding plans that have the property that the optimal bidding plan at each announced price agrees with the continuation of the original plan at that price. This is the prevailing approach taken in the litrature dealing with endogenous change in tastes.[6]

The main result of this paper--that dynamically consistent behavior and expected utility maximizing behavior are equivalent--depends on the particular structure imposed on the decision problem by the rules of English auctions. An examination of the argument reveals that for this result the essential aspect of this structure is that as the bid price increases, the payoff associated with winning declines along a path of distributions that are ordered according to first order stochastic dominance. In the case of English auctions the descending payoff involves leftward translations of the initial distribution. This, however, is not critical and the same result can be obtained using decision trees where the payoff declines along any other path of distributions ordered according to first order stochastic dominance.

A second aspect of our English auctions model that is critical for the main result is that the objects being sold are nondegenerate lotteries. If only degenerate lotteries are allowed, then dynamic consistency is implied by first order stochacstic dominance, but does not imply expected utility maximizing behavior.

Finally, as stated in Proposition 2, dominant English auctions bidding strategies of dynamically consistent decision makers are preference revealing in the sense that decision makers always bid their subjective assessment of the value of the lottery being auctioned. It is interesting to note that preference relations that are preference revealing in the above sense are not necessarily dynamically consistent. Consider, for instance, the case of weighted utility.[7] By the Theorem it is not dynamically consistent. However, the preferences represented by weighted utility are both quasi-concave and quasi-convex, and it can be shown that the optimal English auction bidding strategy of a weighted utility maximizer is preference revealing.[8]

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Footnotes

* We are grateful to Chew Soo Hong for helpful discussions.

1. In English auctions objects are sold according to the following rule: the auctioneer begins by announcing a low openning price, raising the price gradually and terminating the auction as soon as only one bidder remains. The remaining bidder obtains the object at a price equal to the last announced price.

2. The topology of weak convergence satisfies $F_n \longrightarrow F$ iff $F_n(x) \longrightarrow F(x)$ at all continuity points of F. The Levy metric is defined by

$$d(F,G) = \inf\{\epsilon > 0 \mid \text{for all } x \;\; G(x - \epsilon) - \epsilon \leq F(x) \leq G(x + \epsilon) + \epsilon\}.$$

3. A path is a function $H_{(.)}$ that maps [0, 1] into L. Let P be the set of all paths such that $\frac{\partial}{\partial\alpha}H_\alpha(x)$ exists for all $x \in \mathbb{R}$. V is said to be Hadamard differentiable if $\exists\, u : \mathbb{R} \times L \rightarrow \mathbb{R}$, bounded and continuous in its first argument, such that for all $H_{(.)} \in P$,

$$V(H_\alpha) - V(H_0) = \int_X u(x,H_0)d(H_\alpha - H_0) + o(\alpha).$$

u is called the <u>local utility function</u> of V at H_0. For more details see Hadamard [1923].

4. See Karni and Safra [1986].

5. This observation is not specific to the problem at hand. In the context of intertemporal optimization over consumption streams analyzed by Strotz [1956] and Pollak [1968] dynamic inconsistency implies choosing the first period consumption optimally as a part of a consumption plan that the consumer knows he will not follow.

6. A more elaborate exposition of this possibility is beyond the scope of the present paper. The reader is referred to Peleg and Yaari [1973] for a detailed discussion of this idea.

7. See Chew [1981] and Fishburn [1982] for details.

8. For a detailed discussion see Karni and Safra [1986].