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ABSTRACT

In this paper the ability to sign supply-side option value is studied. The compensating and equivalent option prices are defined, and it is argued that equivalent option price is the preferred welfare measure. In the absence of income risk, if the probability distribution of supply is degenerate either with or without the project, one-way test of project acceptance can be established.

ON SUPPLY SIDE OPTION VALUE

I. Introduction

A large literature addresses the problem of measuring welfare change under uncertainty and the use of these measurements in the evaluation of public investments. Two of the central concerns in this literature are i) identifying the correct welfare measure, and ii) establishing the difference between the preferred measure and those that either typically are used in applications or are more readily obtained.

One version of this issue has been much-studied. The preferred measure is taken to be the maximum ex-ante sure payment which equates expected utility in the new situation to expected utility in the initial situation.[1] This payment is known as option price. However, it is supposed that in applications the Hicksian surplus is defined ex-post for each state of nature and then the expected value of these surpluses is employed as the ex-ante welfare measure. Research then is directed to establishing the sign of the difference between the option price and the expected value of surplus. This difference is called option value (OV).[2]

Investigators in this area have established that the ability to determine analytically the sign of option value. If uncertainty concerns the preferences of individuals (i.e., preferences are state-dependent) then the sign of option value cannot be established without putting restrictions on preferences [1, 2, 10, 13, 19, 20].[3] Bishop [2] subsequently determined that if the uncertainty concerns the supply of the good in question, then it is possible to establish that the sign of option value is positive. This result also was derived by Brookshire, et al. [3].

Freeman [11] has pointed out that the Bishop scenario is only one of a number of potential situations of supply-side uncertainty. In general, there may be uncertainty concerning the supply of the good both with and without the project. The Bishop case restricts the supply to be certain if the project is implemented, while there is supply uncertainty if it is not. Freeman claims that if there is uncertainty if the project is implemented but no uncertainty if it is not, the sign of OV is indeterminate. This result was expanded upon by Plummer [18], and embodied as well in Hartman and Plummer [13]. The main purpose of this paper is to clarify these analyses, which are somewhat incomplete.

The option price (OP) is analogous to Hicksian willingness to pay measures of welfare change under certainty, the compensating variation (CV) or the equivalent variation (EV). Like the Hicksian measures, OP comes in two forms, compensating option price (COP) and equivalent option price (EOP). The EV or CV measures can be defined ex-post for each state of nature and then their expected values compared to the corresponding option price. Thus, option value also comes in two forms. Which of these two measures is preferred?

In the case of certainty it has been argued that compensating variation suffers from an intransitivity [5, 14] and hence that equivalent variation is the preferred welfare measure in general, with the two giving the same results if there are only two possible outcomes to be ranked. The same is true for option prices. That is, if there are more than two projects being considered (including the status-quo), the compensating option price is intransitive. Thus, with several projects attention should be restricted to the equivalent option price. However, in this case just knowing the sign of equivalent option value does not allow one to rank the several projects. Thus, the question of the sign of option value only is of interest if there are two projects, in which case knowing the sign of either equivalent or compensating option prices potentially is useful.

Exactly which version is being used in the literature and why sometimes is unclear. Here, we forward definitions of equivalent and compensating option value and show that as long as there is degeneracy in the probability distribution of supply either with or without the project, one of these may be signed without restricting preferences.

For completeness, we also provide some results that have been demonstrated elsewhere in the literature regarding when Marshallian, Hicksian, equivalent, and compensating measures coincide. This makes our paper quite analogous to the certainty literature in which the preferred measure is equivalent variation and one wishes to know when other measures can be used in its stead. Here, the preferred measure is equivalent option price and we provide analysis of when alternative measures can be used "without apology."

II. Welfare Measures under Uncertainty

The planner has a set Δ of feasible projects, $\delta \epsilon \Delta$, and seeks to evaluate them using a benefit-cost approach. To model supply-side uncertainty, we assume that the services of the projects are stochastic. The set of states of nature is S, with $s \epsilon S$. For convenience, we assume that S is finite and let π_s be the exogenous known probability that state s obtains. The service provided by the projects is summarized by a map $q: \Delta x S \rightarrow R$, i.e., $q(\delta, s)$ is the "output" of project δ in state s. The service provided by the projects is a public good which, once the project is implemented, is provided free of charge to individuals. It might be thought of as environmental quality. Of course, individuals do have to pay for the projects.

Individuals choose a bundle of consumption goods $x \epsilon R^n$. This choice takes place ex-post after the state of nature is known. The individuals ordinal preferences are defined over the space of triples $(x, q(\delta, s), s)$. This implies that preferences over alternative consumption bundles and project outputs depend on the state of nature. For each $s \epsilon S$, we assume that these preferences are representable by a state-dependent utility function $U(x, q(\delta, s), s)$ which is increasing in both x and q, twice continuously differentiable, and strictly quasi-concave for each s.[4]

We assume that prices and non-wage incomes are exogenous. Prices, given by $p \epsilon R^n_{++}$ are known constants but income, y(s), is state-dependent. Ex-post, the individual faces the budget set

$$B(s) = \{x \epsilon R^n : p \cdot x \leq y(s), x \geq 0\},$$

which is assumed to be non-empty for all $s \epsilon S$. The agent's state-dependent indirect utility function is defined by:

$$V(p, y(s), \delta, s) = \max\{U(x, q(\delta, s), s) : x \epsilon B(s)\}. \tag{1}$$

It is assumed that $V(\cdot)$ is concave in income (individuals are income risk averse). While choices of x may be made ex-post by individuals, projects and how they will be paid for must be decided upon ex-ante by the planner. It is possible that state-dependent payments for projects can be arranged. Let $w = (w_1, \ldots, w_S)$ be the vector of payments for projects to an individual in each state. For each pair (δ, w), the probability distribution on states of nature induces a discrete probability distribution on levels of utility achieved. We assume that alternative projects are evaluated by individuals according to this induced conditional (on(δ, w)) distribution and further that preferences over alternative distributions satisfy the expected utility hypothesis. Hence, the project-finance scheme pair (δ, w) is preferred to the pair (δ°, w°) if and only if

$$\Sigma_{s \epsilon S} \ \pi_s V(p, y(s)+w(s), \delta, s) > \Sigma_{s \epsilon S} \ \pi_s V(p, y(s)+w^\circ(s), \delta^\circ, s) \qquad (2)$$

For convenience, we define

$$J(\delta, w) = \Sigma_{s \epsilon S} \ \pi_s V(p, y(s)+w(s), \delta, s). \qquad (3)$$

Let the status-quo project be $\delta^\circ \epsilon \Delta$, i.e., δ° is the project in which nothing is done; naturally, $w^\circ = 0$.[5] To analyze alternative payment schemes for projects, Graham [12] introduces the willingness-to-pay locus; we note that a willingness-to-accept (WTA) locus can be defined as well. Formally, the WTP locus is defined by:

$$W(\delta) = \{w \epsilon R^S : J(\delta, w) = J(\delta^\circ, 0)\},$$

while the WTA locus is given by·

$$A(\delta) = \{w \epsilon R^S : J(\delta, 0) = J(\delta^\circ, w)\}.$$

Thus, the locus $W(\delta)$ is the upper envelope of all payments by an individual to obtain the project which yield at least as much expected utility as does the status quo. Similarly, $A(\delta)$ is the lower envelope of payments to individuals in the status quo which leave the person at least as well off as does having the project. Since the WTA locus uses the status-quo as a base, it is natural to call payments schemes along it equivalent payments, while those along the WTP locus, since they use the situation with the project as a base, naturally are compensating payments. This accords with the definitions under certainty [14].

Three elements of $W(\delta)$ and of $A(\delta)$ are of particular interest here: the option price, the surplus point, and the fair bet point. The option price is the point at which the same payment is made in each state. Naturally, both compensating and equivalent option prices exist and are defined by the sure payments

$$COP(\delta) = \{w: w \epsilon W(\delta) \text{ and } w_i = w_j \text{ for all } i,j \epsilon S\}$$

$$EOP(\delta) = \{w: w \epsilon A(\delta) \text{ and } w_i = w_j \text{ for all } i,j \epsilon S\}.$$

The surplus point is the vector of payments with elements corresponding to the Hicksian welfare measure in each state. Hence, we define compensating variation and equivalent variation for project δ in state s by

$$V(p,y(s)-CV_s(\delta),\delta,s) = V(p,y(s),\delta^\circ,s)$$

$$V(p,y(s),\delta,s) = V(p,y(s)+EV_s(\delta),\delta^\circ,s).$$

The expected values of these surplus points are given by:

$$ECV(\delta) = \pi \cdot CV(\delta)$$

$$EEV(\delta) = \pi . EV(\delta)$$

where $\pi = (\pi_1, \ldots, \pi_S)$.

What point on the WTP locus should be used in project evaluation? The answer to this question is quite complicated and depends on a number of factors. A complete discussion of this issue is beyond the scope of this paper [see 7, 12, 17]. Suffice it to say here that in a wide class of problems the option price is either preferred or a useful second-best measure of welfare change.[6] In any event, it is a much-studied one. But which is more reasonable, the EOP or the COP?

If there is more than one project other than the status-quo, then the COP measure is not a valid measure of welfare change. As shown by Hause in a deterministic setting, the compensating variation measure may not rank projects in accord with individual preferences when more than two projects are to be compared. This result holds in the current setting as can be seen using certainty equivalents. Formally, we state

Theorem 1. When Δ contains more than two projects, COP is not a valid measure of welfare change for general preferences.

Proof: Pick any $s \epsilon S$ and define certainty equivalent quality for project δ by the $Q(\delta, w, s)$ satisfying

$J(\delta, w) = \max\{u(x, Q(\delta, w, s), s) : x \epsilon B(s)\} = V(y, (s), Q(\delta, w, s), s)$.

Then by definition, for each $\delta \epsilon \Delta$,

$$V(y(s), Q(\delta, COP(\delta), s) = J(\delta^{\circ}, 0). \qquad (10)$$

Since, the LHS of equation 10 is an indirect utility function, arguments in Hause established that COP is a valid measure of welfare change only for binary rankings unless preferences are restricted. Q.E.D.

In the certainty case, it is known that the compensating variation and equivalent variation are equal (and equal to Marshallian surplus as well) if there are no income effects. In this situation, the problem of the CV not ranking projects correctly disappears. A similar result can be derived here. If preferences are such that the expectation of the marginal utility of income is independent of income and the quality variable, then the COP and EOP are equal. We state that in the following theorem, a proof of which is given in the Appendix. In the sequel we let $Ef(\cdot s) = \Sigma_{s \epsilon S} \pi_s f(\cdot, s)$.

Theorem 2. If $EV_y(y(s),\delta^*,s) = EV_y(y(s),\delta^\circ s)$ then COP$\leq$EOP. If

$EV_y(y(s)\text{-}COP,\delta^*,s) = EV_y(y(s) + EOP,\delta^\circ,s)$ then COP$\geq$EOP.

Both predicates of this theorem will be satisfied when the indirect utility function takes the state-dependent quasi-linear form $V(y(s),q(\delta,s),s) = \alpha_s y(s) + \phi(q(\delta,s),s)$, where $\phi(\cdot)$ is concave in q for each $s\epsilon S$.

We now turn to the problem of establishing a relationship between the magnitude of option price, which may be difficult to discover, and the expected value of the Hicksian surplus measures, which may more readily be obtained.

III. Option Prices and Option Values

When can one deduce useful information for choosing among projects when option price is desired, but only ex-post Hicksian welfare measures are available? Theorem 1 demonstrates that discussion of the relationship between COP and ECV is relevant only if there is one project in addition to the status-quo. In this case, one might seek conditions under which the sign of COP could be inferred from the sign of ECV. But if there is more than one project, unless the postulates of Theorem 2 are involved, one should concentrate on EOP and conditions under which information about EOP's can be gleaned from magnitudes of the EEV's for the various projects.

Unfortunately, when there are several projects, it is apparent that nothing can be inferred in general about the rankings of projects according to EOP from a ranking by EEV. One needs to establish that

$$EEV(\delta^i) \geq EEV(\delta^j) \Rightarrow EOP(\delta^i) \geq EOP(\delta^j)$$

for all projects δ^i and δ^j in Δ. In general, no such implications can be drawn. One needs some information on the <u>magnitudes</u> of differences between

EOP and EEV, not just their signs, across projects. So that knowing the sign of option value potentially is helpful, we restrict our attention to a planning problem with only one project other than the status-quo.

Previous analyses have established that when preferences are state-dependent, option price can be greater or less than expected surplus. The primary problem with assuming state-dependent preferences for the signing of option value is that the marginal utility of income varies across states. To circumvent this problem, two further restrictions on the model are needed. First, we assume that income is state-dependent, i.e., that $y(s) = y$ for all $s \epsilon S$. Second, we focus attention on a situation in which all risk comes from the supply side, i.e., via the function $q(\delta, s)$, with the indirect utility function $V(\cdot)$ not directly dependent on the state of nature.

Following Bishop [2] and Freeman [11], we consider two special cases concerning supply-side risk. In the first case, the status-quo is risky in terms of the quality variable, but implementation of the project eliminates this risk by providing a sure, desirable outcome. For this case, we show in Theorem 3 that COP $\geq$ ECV holds; further, if the marginal utility of income is independent of the quality variable at $(y+EOP,\ q(\delta^{\circ}, s))$, then EOP$\leq$EEV holds as well. In the second case, the status-quo is non-stochastic, but the project δ^{*} provides a risky expected improvement. For this case, EOP$\leq$EEV holds and if the marginal utility of income is independent of quality at $(y-COP,\ q(\delta^{*}, s))$, then COP$\geq$ECV also holds, as stated in Theorem 4. These theorems are proven in the Appendix.

Theorem 3. Assume $y(s)=y$ and that $q(\delta^{*}, s)=q^{*}$ for all $s \epsilon S$. Then COP$\geq$ECV. If in addition

$V_y(y+EOP,q(\delta°,s),s)$ is state-independent, then $EOP \leq EEV$.

Theorem 4. Assume $y(s)=y$ and that $q(\delta°,s)=q°$ for all $s \epsilon S$. Then $EOP \leq EEV$. If in addition $V_y(y-COP,q(\delta^*,s),s)$ is state-independent, then $COP \geq ECV$.

Theorem 2 requires that, for each state $s \epsilon S$, the marginal utility of income is constant. Theorems 3 and 4 do not require constancy of the marginal utility of income, but do require it to be state-independent to reach the stronger conclusions of their second predicates. If both constancy and state-independence are assumed, then one can demonstrate the equivalence of all the relevant measures. Formally, we have

Theorem 5. If the marginal utility of income is a state-independent constant, then $ECV=COP=EOP=EEV$ irrespective of the manner in which income or the function $q(\delta,s)$ depend on the state of nature.

We have established several results that we feel basically to be negative in nature. Unless strong restrictions are placed on preferences, or on the number of projects being evaluated and the manner in which income and the output of the project vary across states, no useful information is contained in the expected value of Hicksian surplus measures if, in fact, option value is the desired welfare measure.

The above discussion concerned the possibility of obtaining useful information from the sign of option value. If something can be said about its magnitude, then some of the ambiguity in ranking projects by expected Hicksian surplus potentially can be resolved. This is not unlike the research of Willig [23] and others regarding the potential magnitude of the difference between Hicksian and Marshallian welfare measures. The main result available here is that a bound for the size of option value can be derived if the range of the variation in the marginal utility of income across states is known. These results were derived by Feenberg and Mills [9] and are stated here to make the discussion self-contained. In particular, let

$$M(y) = \sup\{V_y(y(s),\delta^\circ,s) : s\in S, \delta\in\Delta\}$$

and

$$m(y) = \inf\{V_y(y(s),\delta^\circ,s) : s\in S, \delta\in\Delta\}.$$

We reproduce in the Appendix the proof by Feenberg and Mills [9] of:

Theorem 6. Let $m(\cdot)$ and $M(\cdot)$ be defined as above. Then

$$\left[\frac{m(y(s)+EV(\delta^\circ,s))}{M(y(s)+EV(\delta^\circ,s))} - 1\right] EEV \le EOP \le EEV \le \left[1\ \frac{M(\psi(\sigma)+EOP}{m(y(s)+EOP} - 1\right] EEV.$$

Clearly, Theorem 6 provides an alternative proof of Theorem 5, since in that case, m=M and EOP=EEV (and similarly for COP and ECV). As well, we see that Theorem 6 does not depend in any way on restrictions of either y(s) or $q(\delta,s)$. But it does require significant knowledge of the indirect utility function. Naturally, if one knows $V(\cdot)$ completely, no approximations are necessary. But it may be that the analyst has some information about the magnitude of the marginal utility of income across states even if $V(\cdot)$ itself is not known.

V. Discussion

In this paper, we have provided several results on measurement of welfare change and project evaluation under uncertainty. In many situations, the appropriate welfare change measure is the maximum ex-ante sure payment which equates expected utility in the status-quo to expected utility with implementation of the project. This is the natural stochastic analog of Hicksian equivalent variation. As with the Hicksian measure, considerable attention has been given to conditions under which knowledge of a more readily observed entity, expected Marshallian surplus, can provide insight into choices among alternative projects. Our results basically are negative: unless strong restrictions are placed on preferences, very little insight can be gained.

Obviously, more attention should be given to the problem of assessing option prices. Two basic strategies for estimating option prices exist. The first of these is direct elicitation via sample survey techniques. Such contingent valuation investigations, if carefully conducted, are able to uncover individual option prices. One of the restrictions on this method is that questions should be structured so as to elicit willingness to pay (Cummings, et al. [8]). For an improvement, this implies a compensating option price measure, and for avoiding a deterioration, it implies an equivalent option price measure. However, we have seen that in the case of multiple projects, the compensating measure is not appropriate. This is potentially a problem for contingent valuation studies. Moreover, it is very difficult to represent adequately various aspects of uncertainty in interview settings. Thus the availability of indirect methods for determining option values would be extremely valuable for validation of direct methods as well as valuable per se.

The second class of methods, the indirect ones, make use of observed behavior to deduce willingness to pay. Thus, areas under demand curves approximate Hicksian surplus measures. Recently, considerable attention has been devoted to avoiding approximations and obtaining exact Hicksian welfare measures from demand information [15, 16]. It may be possible to construct similar procedures in the case of uncertainty where changes induced by projects are represented as changed distributions of relevant random variables. Smith [21] and Chavas, et al. [4] provide efforts along these lines.

FOOTNOTES

1. We may use willingness to pay terminology with the understanding that in some cases the payment can be negative; willingness to accept compensation thereby is included without further comment.
2. Some authors (Graham [12]; Cory and Saliba [7]) argue that the correct ex-ante welfare measure is not option price, but the expected value of the fair-bet point (defined below). However, as a second-best measure, option value has recieved substantial attention in the literature.
3. This result has led some authors [19, 17] to conclude that expected surplus should be used since the direction of bias is unknown. This might have some justification if the bias were random and independent either across individuals for a particular project or, reminiscent of Hicks' defense of the potential Pareto improvement test, across projects for a single individual. But an ambiguous bias is not a random one and we therefore should seek to measure option price rather than ignore option value.
4. An equivalent approach is to define state-contingent commodities such that the same physical good in different states is a different good. With this expanded commodity space preference will have a state-independent representation.
5. We assume that payment schemes are chosen to maximize welfare, i.e., actual payments will be the minimum necessary to cover costs of the project.

6. The fair bet points mentioned above are points where the expected value of the payments is maximized, i.e.,

$$F^W(\delta) = \text{Argmax } \{\pi \cdot w : w \in W(\delta)\}$$

$$F^A(\delta) = \text{Argmax } \{\pi \cdot w : w \in A(\delta)\}$$

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APPENDIX

Proof of Theorem 2: Using the assumed concavity of the indirect utility function in income, for each $s \epsilon S$

$$V(y(s)+EOP,\delta^{\circ},s) \leq V(y(s),\delta^{\circ},s) + V_y(y(s),\delta^{\circ},s)[EOP] \quad (A1)$$

and

$$V(y(s)-COP,\delta^{\circ},s) \leq V(y(s),\delta^{*},s) + V_y(y(s),\delta^{*},s)[-COP] \quad (A2)$$

Taking the expectation of A1 yields

$$EV(y(s)+EOP,\delta^{\circ},s) \leq EV(y(s),\delta^{\circ},s) + EV_y(y(s),\delta^{\circ},s)EOP.$$

By definition, the first term on the RHS of this inequality is equal to $EV(y(s)-COP,\delta^{*},s)$. Making this substitution, rearranging and using the fact that EOP is non-stochastic yields:

$$EV(y(s)+EOP,\delta^{\circ},s) - EV(y(s)-COP,\delta^{*},s) \leq EOP\ EV_y(y(s),\delta^{\circ}). \quad (A3)$$

A similar derivation applied to (A2) provides

$$EV(y(s)+EOP,\delta^{\circ},s) - EV(y(s)-COP,\delta^{*},s) \leq [-COP]\ EV_y(y(s),\delta^{*}),$$

and the first conclusion of the Theorem 2 follows.

The second conclusion of Theorem 2 can be shown in a like manner making use of the inequalities.

$$V(y(s),\delta^{\circ},s) \leq V(y(s)+EOP,\delta^{\circ}.s) + V_y(y(s)+EOP,\delta^{\circ},s)[-EOP]$$

and

$$V(y(s),\delta^{*},s) \leq V(y(s)-COP,\delta^{*},s) + V_y(y(s)-COP,\delta^{*},s)[COP]. \quad Q.E.D.$$

Proofs of Theorems 3 and 4: Concavity of $V(\cdot)$ in income implies that:

$$V(y(s)-CV(\delta^{*},s),\delta^{*},s) \leq V(y(s)-COP,\delta^{*},s) + V_y(y(s)-COP,\delta^{*},s)[COP-CV(\delta^{*},s)] \quad (A5)$$

$$V(y(s)+EV(\delta^{\circ},s),\delta^{\circ}.s) \leq V(y(s)+EOP,\delta^{\circ},s) + V_y(y(s)+EOP,\delta^{\circ},s)[EV(\delta^{*},s)-EOP] \quad (A6)$$

Theorem 3 is proven by substituting $V(y(s),\delta°,s)$ for the LHS of (A5), taking expectations and using the definition of COP to yield:

$$0 \leq E\{V_y(y(s)-COP,\delta^*,s)\}[COP-CV(\delta^*,s)]. \qquad (A7)$$

Under the premises of Theorem 3, $y((s)=y$, $q(\delta^*,s)=q^*$ for all s, and $V(\cdot)$ does not depend on s directly. Then, since COP is state-independent, so is $V_y(\cdot)$ and (A7) reduced to $0 \leq COP-ECV(\delta^*)$. A similar derivation applied to (A6) provides

$$0 \leq E\{V_y(y(s)+EOP,\delta°,s)[EV(\delta,s)-EOP]. \qquad (A8)$$

Again, under the premises of Theorem 4, $q(\delta°,s)=q°$ and $y(s)=y$ for all s, $V(\cdot)$ is independent of s, and EOP is non-stochastic. Then, $V_y(\cdot)$ can be removed from the expectations operator in (A8) to yield $0 \leq ECV(\delta^*)-EOP$.

The second results of these theorems are derived in a like manner using (A5) and (A6). But, in the situation of Theorem 3, $q(\delta°,s)$ is not constant function of s. Hence, to remove $V_y(\cdot)$ from the expectations operator in (A8) requires an assumption that $V_y(y+EOP,\delta°,s)$ is independent of quality and, therefore, of s. A similar result holds in Theorem 4 where $q(\delta^*,s)$ is not constant in s. Thus, $V_y(y-COP,\delta^*,s)$ cannot be removed from under the expectations operator in (A7) unless it is assumed that $V_y(y-COP,\delta^*,s)$ is state-independent.

Proof of Theorem 5: From (A7) and (A8) if $V_y(\cdot)$ is state-independent, then $COP \geq ECV$ and $EOP \leq EEV$. However, a similar derivation applied to the inequalities.

$$V(y(s)-COP,\delta^*,s) \leq V(y(s)-CV(\delta^*,s) + V_y(y(s)-CV(\delta^*,s),\delta^*,s)[CV(\delta^*,s)-COP]$$

and

$$V(y(s)-EOP,\delta^*,s) \leq V(y(s)+EV(\delta^*,s)\delta^*,s) + V_y(y(s)+EV(\delta^*,s),\delta^*,s)[EOP-EV(\delta^*,s)] \qquad (A9)$$

yields EOP $\geq$ EEV and COP $\leq$ ECV when V_y is state-independent. The conclusion of the Theorem 2 completes the result. Q.E.D.

Proof of Theorem 6: Rearrangement of (A9) yields

$$EOP \geq \frac{E[V_y(y(s)+EV(\delta^\circ,s)\delta^\circ,s)\ EV(\delta^\circ,s)]}{EV_y(y(s)+EV(\delta^\circ,s),\delta^\circ,d)}$$

Letting $V_y(y(s)+EV(\delta^\circ,s),\delta^\circ,s)$ take on its minimum value in the numerator (a constant) and its maximum value in the denominator preserves this inequality, whence since these minima and maxima are constant,

$$EOP \geq \frac{m}{M} EEV.$$

A similar derivation using

$$V(y(s)+EV(\delta^*,s)\delta^*,s) \leq V_y(y(s) + EOP,\delta^*,s)+ V_y(y(s)+EOP,\delta^\circ,s)[EV(\delta^*,s)-EOP]$$

yields

$$EOP \leq \frac{E[V_y(y(s)+EOP,\delta^\circ,s)\ EV(\delta^\circ,s)]}{EV_y(y(s)+EOP,\delta^\circ,s)} \leq \frac{M}{m}\ EEV.$$

The conclusion of Theorem 6 follows immediately. Q.E.D.