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Risk -- Mathematical models

RISK AVERSION REVISITED: A CLOSER LOOK AT MEANING AND MEASUREMENT

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ABSTRACT

Contrary to popular belief, risk aversion is not always equivalent to concavity of the "utility of income" function. When this equivalence fails, Arrow-Pratt coefficients are not a meaningful measure of risk aversion. The pivotal issue is whether farmers' choices under risk are entirely determined by their choices under certainty.

1. Introduction

It is commonly believed that, within expected-utility theory, risk aversion is equivalent to the concavity of the utility function (of, say, income). In this paper, we show that this belief is not accurate. Rather, while the equivalence holds (in a weak form) when risk preferences are continuous, it may fail in one direction, and "appear" to fail in the other, when risk preferences are discontinuous. Importantly, the axioms of expected-utility theory do not imply that risk preferences need be continuous (Weiss 1985a, 1986). Furthermore, an assumption of continuous preferences imposes the not-entirely-innocent requirement that behavior under certainty completely determine behavior under uncertainty. Such an assumption is not obvious on any empirical grounds.

Risk aversion is customarily illustrated or even defined through the use of a diagram displaying a utility curve lying in the "income/utility plane." Unfortunately, this graphical approach is misleading, for it only shows the individual's preferences between certainties. In the general case, it conveys no information whatsoever concerning preferences between continuous distributions, such as normals. For example, it is perfectly possible for a farmer's risk preferences to be discontinuous—for his preferences between certainties not to determine his preferences between continuous distributions—even while his utility function of income is continuous. In such a case, concavity is not equivalent to risk aversion. Moreover, the use of Arrow-Pratt "risk aversion coefficients" to measure risk aversion becomes entirely spurious.

In what follows, we shall present a new, more fundamental way of looking at risk aversion. Our purpose is not only to clarify the relationship between risk aversion and concave utility, but, more generally, to contribute to the ongoing search for improved theories of farmers' behavior under risk by placing the key notion of risk aversion on a conceptually clearer and more rigorous foundation.

2. Preliminary Risk Concepts and Definitions

The theory of individual choice under risk is concerned with the choices that individuals make when confronted with alternative risky prospects, called *lotteries* and represented mathematically as cumulative probability distributions.

For each r, the lottery F_r defined by $F_r(t) = 0$ if $t < r$, $F_r(t) = 1$ if $t \geq r$, is called *degenerate. For purposes of empirical application, degenerate lotteries represent "certainties."* A lottery L is called *simple* if it can be expressed as a convex combination, $L = \sum_i p_i F_{r_i}$, of finitely many degenerate lotteries F_{r_i} (where each p_i is nonnegative and $\sum_i p_i = 1$). A lottery is called *continuous* if it is continuous as an ordinary function.

In risk theory, as in demand theory, an essential requirement is that the economic agent's choice set be *convex*. Note that any choice set of lotteries is contained in the set of all numerical-valued functions on the number line; moreover, the latter set is a *vector space* under the usual operations of addition/subtraction of functions and multiplication of functions by numbers. A choice set of lotteries is *convex* if it is convex as a subset of this vector space. Henceforth, we shall call a convex choice set of lotteries a *lottery space*. If L is a lottery space and V a numerical-valued function defined on L, we call V *linear* if

$$V(tL_1 + (1-t)L_2) = tV(L_1) + (1-t)V(L_2)$$

whenever $0 \leq t \leq 1$ and L_1, L_2 are lotteries in L.

The notion of an individual's "risk preferences" is formalized as follows. A (*weak*) *preference ordering*, $\succsim$, on a lottery space $\mathcal{L}$ is a complete, transitive, binary relation on $\mathcal{L}$. Given a preference ordering $\succsim$, we can define the corresponding *indifference relation*, $\sim$, on $\mathcal{L}$ by: $L_1 \sim L_2$ if and only if $L_1 \succsim L_2$ and $L_2 \succsim L_1$. We define the corresponding *strong* preference ordering, $\succ$, on $\mathcal{L}$ by: $L_1 \succ L_2$ if and only if $L_1 \succsim L_2$ but not $L_1 \sim L_2$. If $\mathcal{L}$ is a lottery space and $\succsim$ a preference ordering on $\mathcal{L}$, we call the pair $(\mathcal{L},\succsim)$ a *preference lottery space*. A *utility function*, U, for $(\mathcal{L},\succsim)$ is a numerical-valued function defined on $\mathcal{L}$ such that, for any L_1, L_2 in $\mathcal{L}$, $U(L_1) \geq U(L_2)$ if and only if $L_1 \succsim L_2$. A utility function for $(\mathcal{L},\succsim)$ is called *measurable* if it is linear on $\mathcal{L}$. The concept of a "measurable utility function" is central to expected-utility theory. Of course, the linearity property mimics the computation of an expected utility. *The whole thrust of von Neumann and Morgenstern and Herstein and Milnor was to prove that, under certain plausible conditions, an individual's risk preference ordering can always be represented by a measurable utility function. We stress, however, that a measurable utility function need not take the form of an "expected utility integral."*

Next, we explain what is meant by "continuous risk preferences." Recall that a sequence $\{G_n\}$ of probability distribution functions is said to *converge* to a distribution G if $\lim_{n\to\infty} G_n(t) = G(t)$ for each point t of continuity of G (Breiman, p. 159). Let $\mathcal{L}$ be a lottery space, U a numerical-valued function defined on $\mathcal{L}$, and L a lottery in $\mathcal{L}$. We say U is *continuous at* L if, whenever $\{L_n\}$ is a sequence of lotteries in $\mathcal{L}$ that converges to L, one has $U(L_n) \to U(L)$ as $n \to \infty$. If U is continuous at each lottery in $\mathcal{L}$, we say U is *continuous*. We call a risk preference ordering $\succsim$ *continuous* and speak of "continuous risk preferences" when there exists a continuous utility function representing $\succsim$.

Just as the utility functions of demand theory assign numerical values to commodity bundles, the utility functions of risk theory, as we have seen, assign numerical values to *lotteries*. Where, then, do "utility functions" of income, as displayed in the "income/utility plane" in most treatments of risk aversion, fit in? To answer this, let U be a numerical-valued function defined on a lottery space $\mathcal{L}$ that contains all degenerate lotteries. Then, we can define a numerical function u by $u(r) = U(F_r)$ for each r. We say u is *induced (on the number line) by* U. If, in particular, $(\mathcal{L},\succsim)$ is a preference lottery space for which U is a utility function, we may call u the *utility function induced by* U and interpret it as (for example) a "utility function of income." Note, however, that u reflects only the values of U "under certainty," that is, at degenerate lotteries. Without further assumptions, U cannot be recovered from u. Indeed, it is demonstrated in Weiss (1986) that *any* numerical function, u, proposed as a "utility function of income," can be induced by measurable utility functions representing infinitely many *different* risk preference orderings! Thus, utility functions of income play the role of "value functions" (Schoemaker, pp. 533-35) in the sense that they (in effect) assign utility values to certainties but do not generally determine a complete utility function on $(\mathcal{L},\succsim)$.

Finally, we define the various types of response to risk. Denote the mean of a lottery L by E(L). Let $(\mathcal{L},\succsim)$ be a preference lottery space such that, for each L in $\mathcal{L}$, E(L) is finite and the degenerate lottery $F_{E(L)}$ is also in $\mathcal{L}$. Then, $(\mathcal{L},\succsim)$ (or $\succsim$) is called

(1) *weakly risk-averse* if, for each L in $\mathcal{L}$, $F_{E(L)} \succsim L$,

(2) *risk-neutral* if, for each L in $\mathcal{L}$, $F_{E(L)} \sim L$, and

(3) *weakly risk-loving* if, for each L in $\mathcal{L}$, $L \succsim F_{E(L)}$.

If $\succsim$ is replaced by $\succ$ in (1) or (3), then $(\mathcal{L},\succsim)$ (or $\succsim$) is called *strongly risk-averse* or *strongly risk-loving*, respectively.

The preceding definition makes it clear that risk aversion is a purely ordinal concept.

3. Risk Aversion and Concave Utility

We can now clarify the relationship between risk aversion and concavity of the "utility function of income":

Let $(\mathcal{L},\succsim)$ *be a preference lottery space for which* $\mathcal{L}$ *contains every degenerate lottery and for which every lottery in* $\mathcal{L}$ *has a finite mean. Assume* $(\mathcal{L},\succsim)$ *has a measurable utility function,* U, *that induces on the number line a utility function ("of income"),* u. *Then:*

(1) *If* $\succsim$ *is weakly risk-averse (respectively, weakly risk-loving; risk-neutral), then* u *is concave (respectively, convex; linear). The analogous statement holds if "weakly" is replaced by "strongly," using "strict inequality" notions of concavity and convexity.*

(2) *If* u *is concave (respectively, convex; linear)* and U continuous, *then* $\succsim$ *is weakly risk-averse (respectively, weakly risk-loving; risk-neutral).*

To prove (1), suppose $\succsim$ is weakly risk-averse, and consider any numbers a, b, and p with $0 < p < 1$. Put $L = pF_a + (1-p)F_b$; then (since $\mathcal{L}$ is convex) L is in $\mathcal{L}$ and $E(L) = pa + (1-p)b$. Since $F_{E(L)} \succsim L$, it follows that $U(F_{E(L)}) \geqslant U(L)$, so that, by the definition of u and the linearity of U,

$$u\left[pa+(1-p)b\right] \geqslant pu(a) + (1-p)u(b).$$

Thus, u is concave.

If $\succsim$ is weakly risk-loving, the proof is similar. Finally, if $\succsim$ is risk-neutral, then it is both weakly risk-averse and weakly risk-loving, so that u is both concave and convex, hence linear.

The second half of (1) is proved similarly.

To prove (2), suppose u is concave and U continuous, and consider any L in $\mathcal{L}$. Then (see Weiss 1985b, 1986) there is a sequence $\{L_i\}$ of simple lotteries (hence, elements of $\mathcal{L}$) such that L_i converges to L and $E(L_i) \to E(L)$ as $i \to \infty$. We may write

$$L_i = \sum_{j=1}^{n_i} p_{ij} F_{t_{ij}}$$

for each i. Since u is concave, we have

$$\begin{aligned} U\left[F_{E(L_i)}\right] &= u\left[E(L_i)\right] \\ &= u\left[\sum_{j=1}^{n_i} p_{ij} t_{ij}\right] \\ &\geq \sum_{j=1}^{n_i} p_{ij} u(t_{ij}) \\ &= U\left[\sum_{j=1}^{n_i} p_{ij} F_{t_{ij}}\right] \\ &= U(L_i) \end{aligned}$$

for each i. However, as $i \to \infty$, L_i converges to L and (since $E(L_i) \to E(L)$) $F_{E(L_i)}$ converges to $F_{E(L)}$. Thus, by the continuity of U, $U\left[F_{E(L)}\right] \geq U(L)$, whence $F_{E(L)} \succsim L$. It follows that $\succsim$ is weakly risk-averse.

For the case when u is convex, the proof of the corresponding result is similar. Finally, if u is linear, it is both concave and convex, and the corresponding result follows from the two previous cases.

Assertion (2) raises the question of whether concavity of u implies weak risk averseness for $\succsim$ when U is *not* continuous. We now show that it does not; in fact, not even *strict* concavity of u would suffice.

Toward this end, let u be an arbitrary numerical function. We shall prove that, whether or not u is concave, one can always find a measurable utility function, U, for a *non*-risk-averse preference ordering, $\succsim$, that induces u. To do this, we shall first define certain linear functions U_1, U_2 and then define U in terms of these.

Let $\mathcal{H}$ be the lottery space of all simple lotteries. Define U_1 on $\mathcal{H}$ as follows: given any H in $\mathcal{H}$ (where, say, $H = \sum_{i=1}^{m} p_i F_{r_i}$), put

$$U_1(H) = \sum_{i=1}^{m} p_i u(r_i).$$

It can be shown (see Weiss 1985a, 1986) that U_1 is well-defined (that is, that $U_1(H)$ does not depend on the particular way in which H is represented as a convex combination of degenerate lotteries). Clearly, U_1 is linear. Moreover, it induces u, since $U_1(F_r) = u(r)$ for each r.

Next, let $\mathcal{G}$ be the lottery space of all continuous lotteries with finite mean. Choose any $a > 0$ such that $a > 2u(0)$. Define U_2 on $\mathcal{G}$ by

$$U_2(G) = a[1-G(0)]$$

for each G in $\mathcal{G}$. It is readily verified that U_2 is linear.

Finally, let $\mathcal{L}$ be the convex hull of the union of the sets $\mathcal{H}$ and $\mathcal{G}$, and define U on $\mathcal{L}$ as follows: given any L in $\mathcal{L}$, there exist (by properties of convex hulls) lotteries H in $\mathcal{H}$ and G in $\mathcal{G}$, and a number p satisfying $0 \leq p \leq 1$, such that $L = pH + (1-p)G$. Furthermore, H is unique unless $p = 0$, and G is unique unless $p = 1$ (see Chung, pp. 7–9; Weiss 1983). Thus, the function U on $\mathcal{L}$ defined by

$$U(L) = pU_1(H) + (1-p)U_2(G)$$

is well-defined. Clearly, U is linear and induces u. Moreover, U is a measurable utility function for the risk preference ordering $\succsim$ on $\mathcal{L}$ defined by: $L_1 \succsim L_2$ if and only if $U(L_1) \geq U(L_2)$.

Now, choose any G in $\mathcal{G}$ such that $G(0) = 1/2$ and $E(G) = 0$ (the probability distribution function of some uniform density centered at 0 will obviously do). Then

$$U\left[F_{E(G)}\right] = u\left[E(G)\right] = u(0).$$

On the other hand,

$$U(G) = a[1-G(0)] = a/2 > u(0)$$

by the definition of a. Thus $G > F_{E(G)}$, whence $\succsim$ is not weakly risk-averse. However, since u was arbitrary, we can take it to be strictly concave. We have thus demonstrated that, in the absence of an assumption of continuous preferences, even strict concavity of u does not imply weak risk-averseness for $\succsim$. *Indeed, since* u *was arbitrary, we can conclude that* no *assumption concerning* u *alone is sufficient to guarantee risk aversion. Rather, risk aversion can only be guaranteed by an assumption at the more abstract level of the risk preference ordering itself.*

Assertion (1) establishes that, whether or not risk preferences are continuous, weak risk averseness of $\succsim$ implies that an induced utility function u must be concave. Nevertheless, this implication can still *appear* to fail: for, notwithstanding (1), we shall now construct a weakly risk-averse preference ordering that, over all continuous lotteries corresponding to bounded random variables (for brevity: over all continuous *bounded* lotteries), is represented by an "expected-utility integral" whose integrand is *not* concave. That is, we shall construct a weakly risk-averse $\succsim$ and a *non*-concave numerical function v such that, for any continuous bounded lotteries L_1, L_2, we have $L_1 \succsim L_2$ if and only if

$$\int_{-\infty}^{\infty} v(t)dL_1(t) \geq \int_{-\infty}^{\infty} v(t)dL_2(t).$$

Furthermore, if, for some interval [a,b], we restrict the set of lotteries considered to those arising from random variables taking values only in [a,b], then we can even specify v to be *strictly convex*! This example seems to fly in the face of traditional expected-utility analysis. Yet, it is entirely consistent with the classical expected-utility axioms. It merely reflects a more careful examination of the subject in which the notion of "risk aversion" has been disentangled from that of "continuous preferences."

To accomplish the construction, let *G* be the lottery space of all continuous bounded lotteries, let *H* be as before, and let *L* be the convex hull of the union of the sets *H* and *G*. Given any concave, numerical function u, define a linear function U_1 on *H* by

$$U_1(H) = \int_{-\infty}^{\infty} u(t)dH(t)$$

for each H in *H*. Also, let U_2 be any linear function on *G* satisfying the following property ("Property P"): "For each G in *G*, $u\left[E(G)\right] \geq U_2(G)$." (We give examples below.) Finally, define a linear function, U, on *L* by

$$U\left[pH + (1-p)G\right] = pU_1(H) + (1-p)U_2(G) \quad (0 \leq p \leq 1,\ H \text{ in } \mathcal{H},\ G \text{ in } \mathcal{G})$$

(this definition is justified by the same argument given earlier). Note that U is a measurable utility function for the preference ordering $\gtrsim$ on *L* defined by: $L_1 \gtrsim L_2$ if and only if $U(L_1) \geq U(L_2)$. Furthermore, $\gtrsim$ is weakly risk-averse. In fact, suppose L is in *L*. Then, there exist lotteries H in *H* and G in *G* and a number p satisfying $0 \leq p \leq 1$ such that $L = pH + (1-p)G$. Thus, by the concavity of u and Property P of U_2, and letting X be any random variable whose probability distribution function is H, we have

$$\begin{aligned} U(F_{E(L)}) &= U_1(F_{E(L)}) \\ &= u\left[pE(H) + (1-p)E(G)\right] \\ &\geq pu\left[E(H)\right] + (1-p)u\left[E(G)\right] \\ &\geq pu\left[E(X)\right] + (1-p)U_2(G) \end{aligned}$$

$$\geq pE(u(X)) + (1-p)U_2(G)$$
$$= pU_1(H) + (1-p)U_2(G)$$
$$= U(L).$$

We now exhibit a linear function U_2 that satisfies Property P and allows U on $\mathcal{G}$ to be expressed as the integral of a non-concave function. For this, let v be any continuous numerical non-concave function dominated by u, that is, for which $v(t) \leq u(t)$ for all t. Define U_2 on $\mathcal{G}$ by

$$U_2(G) = \int_{-\infty}^{\infty} v(t)dG(t)$$

for any G in $\mathcal{G}$. Then, for each G in $\mathcal{G}$, letting Y be any random variable whose probability distribution function is G and applying Jensen's Inequality, we obtain

$$u\left[E(G)\right] = u\left[E(Y)\right] \geq E(u(Y)) \geq E(v(Y)) = U_2(G),$$

so that U_2 satisfies Property P. Since $U(G) = U_2(G)$ for all G in $\mathcal{G}$, U has the desired representation as an integral over $\mathcal{G}$. Alternatively, it is clear that, if we had defined $\mathcal{G}$ as the set of all continuous lotteries that arise from random variables whose values all lie in some specified interval [a,b], and if we had required that u dominate v only over [a,b], then we could have chosen v to be strictly convex.

4. Implications for Identifying and Modeling Risk Aversion

Consider an investigator who wishes to construct an economic model involving risk-averse behavior. Within the *traditional* interpretation of expected-utility theory (which, we have noted, does not follow from the expected-utility axioms), he would, in effect, assume that measurable utility functions take the form of expected-utility integrals, and he would adopt a concave numerical function (a "von Neumann-Morgenstern utility function") as a "generator" of the risk behavior. Similarly, if he were attempting to *estimate* a utility function underlying presumedly risk-averse behavior, he could adopt a parameterized class of concave functions and attempt to obtain econometric estimates of the "true" parameters of a von Neumann-Morgenstern utility function. These approaches, however, carry the implicit assumption that the individuals under study have the same "frames of reference" toward certainty (degenerate lotteries) and "continuous uncertainty" (continuous lotteries), since, in both cases, the measurable utility of a lottery is determined, through the integral formula, by the utility values assigned to certainties. If, on the contrary, the researcher does not wish to rule out *a priori* the possibility of different frames of reference—the possibility that an individual's risk preferences may be different for certainties and for continuous lotteries, so that his risk preferences are discontinuous—then the assumption of concavity for v would be an untenable restriction that would preclude either identifying or modeling certain risk-averse behavior; for, as our first example proved, a measurable utility function for a risk-averse preference ordering can induce a *non*-concave utility function of income. In such a case, no *concave* utility function can be induced by any measurable utility function for the same preference ordering (see Weiss 1986).

On the other hand, our second example suggests that some risk-averse preference orderings might manifest the *illusion* of being determined by a non-concave—or even strictly convex—utility function of income; for, as we established in that example, preferences between certain continuous lotteries can be determined by such an "apparent" utility function even when the full preference ordering is risk-averse. Interestingly, Hildreth and Knowles, working within the expected-utility framework, cite several empirical studies (including their own) of individuals' risk preferences that produced apparently risk-neutral or risk-loving responses (in effect, "non-concavities" in the utility function of income) in cases where one might expect the decisionmakers' "true" preferences to be risk-averse (Hildreth and Knowles, p. 33). They characterize these non-concavities as inaccurate and suggest various possible explanations for their occurrence (see also Robison). The theory that we have delineated here, however, provides an alternative hypothesis: the non-concavities are legitimate; interpreted in terms of measurable utility functions, they do not contradict risk-averseness. Rather, the respondents in the studies, while being risk-averse, may have had discontinuous, "two part" measurable utility functions (reflecting dichotomous behavior toward certainty and "continuous uncertainty") and may have responded to some queries in the studies as though they were using their "continuous uncertainty" rule (U_2, in the notation of our second example, with v non-concave) rather than their "certainty" rule (U_1, with u concave). This question, along with the more general question of whether farmers' behavior under uncertainty may not be entirely determined by their behavior under certainty (resulting in discontinuous risk preferences and inapplicability of Arrow-Pratt risk aversion coefficients), would appear to merit further investigation by agricultural economists.

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