Statistics

Netherlands School of Economics

ECONOMETRIC INSTITUTE

Report 7105

PREDICTION IN THE GENERAL MULTIPLICATIVE MODEL; AN APPLICATION TO AUTOCORRELATED DISTURBANCES

by R. Teekens and J. Koerts

June 10, 1971

Preliminary and Confidential

PREDICTION IN THE GENERAL MULTIPLICATIVE MODEL; AN APPLICATION TO AUTOCORRELATED DISTURBANCES

by R. Teekens and J. Koerts

Contents

1. THE GENERAL MULTIPLICATIVE MODEL

1.1. Introduction

The multiplicative model with constant elasticities can be written as

(1.1) $$Y_i = \prod_{k=1}^{K} Z_{ik}^{\delta_k} v_i \qquad i = 1, \ldots, N$$

or after the log transformation:

(1.2) $$y = X\beta + u$$

with $[y_i] = [\ln Y_i]$
$[x_{ik}] = [\ln z_{ik}]$
$[\beta_k] = [\delta_k]$
$[u_i] = [\ln v_i]$

We shall now consider the estimation of the mean of the dependent variables under the following assumptions on the vectors u and v.[1] Let

(1.3) $$E(v) = \iota$$

and

(1.4) $$\text{var } v = \Omega = [\omega_{ij}]$$

Then

(1.5) $$E(u) = \mu$$

(1.6) $$\text{var } u = \Sigma = [\sigma_{ij}]$$

where the elements of μ and Σ are functions of the elements of Ω, the character of the relationship will depend on the assumption concerning the distribution of v.

To find the minimal M.S.E. estimator of

(1.7) $$\eta(x) = E[Y(x)] = \exp\{x'\beta\}$$

for known variance we generalize the approach followed in the case of a scalar variance covariance matrix of v. The first task is to find the generalised least squares estimator of β. There is, however, a complication in this problem since the mean of u is not equal to zero. To overcome this problem we apply the following transformation to model (1.2):

Let

(1.8) $$\begin{cases} y^* = y - \mu \\ u^* = u - \mu \end{cases}$$

then in the transformed model

(1.9) $$y^* = X\beta + u^*$$

---

[1] For an introduction to this subject we refer to Teekens and Koerts [1970 , a] and [1970 , b], where the same problem has been considered under more restrictive assumptions.

u^* has zero mean and variance covariance matrix as in (1.6). In model (1.9) the generalized l.s. estimator of β equals

(1.10) $$\hat{\beta}^* = X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y^*$$

This estimator is substituted in the estimator of $\eta(x)$:

(1.11) $$Y_x = c.e^{x'\hat{\beta}^*}$$

where c remains to be specified. If we minimize the M.S.E. of Y_x with respect to c, we obtain the minimal M.S.E. estimator of $\eta(x)$:

(1.12) $$\bar{Y}^*_x = \frac{M_{u^*}(\ell_x)}{M_{u^*}(2\ell_x)}\, e^{x'\hat{\beta}^*}$$

where $\ell'_x = x'(X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}$ and where $M_{u^*}(\ )$ stands for the moment generating function of the vector u^*.

It is worth noting that this estimator of $\eta(x)$ is equivalent to the estimator which would result if we ignore the fact that in model (1.2) the vector of disturbances does not have zero mean, i.e. if we substitute the generalized l.s. estimator of model (1.2)

(1.13) $$\hat{\beta} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y$$

into the estimation function of $\eta(x)$:

(1.14) $$Y_x = c.e^{x'\hat{\beta}}$$

In that case minimization of the M.S.E. of Y_x with respect to c gives us

(1.15) $$\bar{Y}_x = \frac{M_u(\ell_x)}{M_u(2\ell_x)}\, e^{x'\hat{\beta}}$$

The equivalence of $\bar{Y}^*_x$ and $\bar{Y}_x$ can be proved as follows. Let us first consider the m.g.f. of u^*. As can be seen from (1.8) u^* is a linear function of u, hence

(1.16) $$M_{u^*}(\tau) = E[e^{u^{*\prime}\tau}] = E[e^{u'\tau-\mu'\tau}] = e^{-\mu'\tau}M_u(\tau)$$

Moreover, $\hat{\beta}^*$ can be written as a function of $\hat{\beta}$, for

(1.17) $$\hat{\beta}^* = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y^* = \hat{\beta} - (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}\mu$$

hence

(1.18) $$\exp\{x'\hat{\beta}^*\} = \exp\{x'\hat{\beta} - \ell_x'\mu\}$$

If we apply (1.16) and (1.18) to (1.12) we obtain

(1.19) $$\bar{Y}_x^* = \frac{e^{-\ell_x'\mu} M_u(\ell_x)}{e^{-2\ell_x\mu} M_u(2\ell_x)} e^{x'\hat{\beta}-\ell_x'\mu} = \bar{Y}_x$$

Hence, we can use (1.15) as an estimator of $\eta(x)$, and it is not necessary to apply transformation (1.8) to obtain the optimal estimator.

1.2. Application to the Lognormal Case

Assume that in model (1.1) the disturbances are lognormally distributed with mean and variance covariance matrix as specified in (1.3) and (1.4) respectively. Then the vector of disturbances in the transformed model (1.2) has a multivariate normal distribution with mean and variance covariance matrix as given in (1.5) and (1.6):

(1.20) $$u \sim N(\mu, \Sigma)$$

The estimator of $\eta(x)$ follows from substitution of the moment-generating function of u

(1.21) $$M_u(\tau) = \exp\{\tau'\mu + \tfrac{1}{2}\tau'\Sigma\tau\}$$

into (1.15). Then we get:

(1.22) $$\bar{Y}_x = \exp\{x'\hat{\beta} - (\ell_x'\mu + \tfrac{3}{2}\ell_x'\Sigma\ell_x)\}$$

In order to trace the implications of assumption (1.3) on μ, we follow a reverse solution. Starting from (1.20), the expectation of v_i and $v_i v_j$ can easily be found by making use of the m.g.f. of u. From (1.21) it follows that

$$(1.23) \qquad E(v_i) = E[e^{u_i}] = \exp\{\mu_i + \tfrac{1}{2}\sigma_{ii}\} \qquad i = 1, \ldots, N$$

and

$$(1.24) \qquad E(v_i v_j) = E[e^{u_i+u_j}] = \exp\{\mu_i + \mu_j + \tfrac{1}{2}(\sigma_{ii} + 2\sigma_{ij} + \sigma_{jj})\} \qquad i, j = 1, \ldots, N$$

It can now be seen that assumption (1.3) implies that

$$(1.25) \qquad \mu_i = -\tfrac{1}{2}\sigma_{ii} \qquad i = 1, \ldots, N$$

Substitution of (1.25) into (1.24) yields:

$$(1.26) \qquad E(v_i v_j) = \exp\{\sigma_{ij}\} \qquad i, j = 1, \ldots, N$$

Hence

$$(1.27) \qquad \Omega = \begin{bmatrix} e^{\sigma_{11}} & \ldots & e^{\sigma_{1N}} \\ \vdots & & \vdots \\ e^{\sigma_{N1}} & \ldots & e^{\sigma_{NN}} \end{bmatrix} - \begin{bmatrix} 1 & \ldots & 1 \\ \vdots & & \vdots \\ 1 & \ldots & 1 \end{bmatrix}$$

or, to put it the other way around, let

$$(1.28) \qquad \omega_{ij} = e^{\sigma_{ij}} - 1 \qquad i, j = 1, \ldots, N$$

Then

$$\sigma_{ij} = \ln (\omega_{ij} + 1) \qquad i, j = 1, \ldots, N \tag{1.29}$$

and (1.25) becomes

$$\mu_i = -\tfrac{1}{2} \ln (\omega_{ii} + 1) \qquad i = 1, \ldots, N \tag{1.30}$$

Therefore, if v_i is lognormally distributed with mean equal to the unit vector and with a variance covariance matrix Ω, u is multivariate normally distributed with mean and variance covariance matrix as defined in (1.30) and (1.29) respectively, and the estimator Y_x as given in (1.22) is completely specified.

2. AUTOCORRELATION OF THE TRANSFORMED DISTURBANCES

2.1. Introduction

In this chapter we investigate the multiplicative model where the transformed disturbances u_i follow a first order Markov scheme:

$$u_i = \rho u_{i-1} + \varepsilon_i \qquad i = 1, \ldots, N \tag{2.1}$$

in which $|\rho| < 1$ and in which the ε_i's are independently normally distributed with mean μ_ε and variance σ_ε^2. First we trace the implications of the assumption

$$Ev_i = 1 \qquad i = 1, \ldots, N \tag{2.2}$$

on the expectation of ε_i. From (2.1) it can easily be seen that

$$v_i = v_{i-1}^{\rho} . e^{\varepsilon_i} \qquad i = 1, \ldots, N \tag{2.3}$$

We may now express v_{i-1} in terms of v_{i-2} and $e^{\varepsilon_{i-1}}$, etc. then we get

(2.4) $$v_i = \prod_{j=0}^{\infty} e^{\rho^j \varepsilon_{i-j}}$$

Since the ε_i's are independently distributed we can write

(2.5) $$Ev_i = \prod_{j=0}^{\infty} E[e^{\rho^j \varepsilon_{i-j}}]$$

and it can easily be seen that

(2.6) $$E[e^{\rho^j \varepsilon_{i-j}}] = \exp\{\rho^j \mu_\varepsilon + \tfrac{1}{2}\rho^{2j}\sigma_\varepsilon^2\}$$

which is the m.g.f. of ε_{i-j} evaluated in the point ρ^j. Hence

(2.7) $$Ev_i = \prod_{j=0}^{\infty} \exp\{\rho^j \mu_\varepsilon + \tfrac{1}{2}\rho^{2j}\sigma_\varepsilon^2\}$$

Taking the logarithm at both sides of (2.7), we get

(2.8) $$\ln[Ev_i] = \sum_{j=0}^{\infty} \rho^j \mu_\varepsilon + \sum_{j=0}^{\infty} \tfrac{1}{2}\rho^{2j}\sigma_\varepsilon^2$$

$$= \frac{\mu_\varepsilon}{1-\rho} + \frac{\tfrac{1}{2}\sigma_\varepsilon^2}{1-\rho^2}$$

Thus, assumption (2.2) is fulfilled if

(2.9) $$\frac{\mu_\varepsilon}{1-\rho} = \frac{-\tfrac{1}{2}\sigma_\varepsilon^2}{1-\rho^2}$$

or

(2.10) $$\mu_\varepsilon = \frac{-\tfrac{1}{2}\sigma_\varepsilon^2}{1+\rho}$$

This being established we are now able to derive the mean and the variance covariance matrix of the vector u. From (2.1) we deduce:

(2.11) $$u_i = \sum_{j=0}^{\infty} \rho^j \varepsilon_{i-j}$$

Hence

$$\text{(2.12)} \qquad Eu_i = \sum_{j=0}^{\infty} \rho^j E(\varepsilon_{i-j}) = \frac{-\frac{1}{2}\sigma_\varepsilon^2}{1 - \rho^2}$$

and

$$\text{(2.13)} \qquad E[(u_i - Eu_i)(u_{i-\ell} - Eu_{i-\ell})] =$$

$$E[\{\sum_{j=0}^{\infty} \rho^j(\varepsilon_{i-j} - E\varepsilon_{i-j})\}\{\sum_{j=0}^{\infty} \rho^j(\varepsilon_{i-\ell-j} - E_{\ i-\ell-j})\}]$$

$$= E[\{\sum_{j=0}^{\infty} \rho^j \delta_{i-j}\}\{\sum_{j=0}^{\infty} \rho^j \delta_{i-\ell-j}\}]$$

where

$\delta_k = \varepsilon_k - E\varepsilon_k$

$$= \sum_{j=\ell}^{\infty} \rho^{2j-\ell} E(\delta_{i-j}^2) = \frac{\rho^\ell}{1 - \rho^2}\sigma_\varepsilon^2 \qquad \ell = 0, \ldots, N-1$$

where use has been made of $E\delta_i\delta_j = 0$ for $i \neq j$. Hence the covariance matrix of the vector u equals

$$\text{(2.14)} \quad \Sigma = \frac{\sigma_\varepsilon^2}{1 - \rho^2} \begin{bmatrix} 1 & \rho & \cdots & \rho^{N-1} \\ \rho & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho \\ \rho^{N-1} & \cdots & \rho & 1 \end{bmatrix} = \sigma^2 \begin{bmatrix} 1 & \rho & \cdots & \rho^{N-1} \\ \rho & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho \\ \rho^{N-1} & \cdots & \rho & 1 \end{bmatrix} = \sigma^2 P$$

where $\sigma^2 = \sigma_\varepsilon^2/(1 - \rho^2)$ and its mean (see 2.12) is

$$\text{(2.15)} \qquad \mu = -\tfrac{1}{2}\sigma_\varepsilon^2/(1 - \rho^2).\iota = -\tfrac{1}{2}\sigma^2.\iota$$

2.2. Prediction

When dealing with prediction of the dependent variable in a model with independently distributed errors, we used $\bar{Y}_x$, a minimal MSE estimator of $\eta(x)$. If the errors show autocorrelation, $\bar{Y}_x$ is not appropriate as a predictor, since in that case we should use the information which is given by the sample concerning the values of the errors over the sample period. Because of the dependency of the errors, they give some information concerning subsequent values of the errors.

Let us formalize this reasoning and let us consider a prediction of the dependent variable in the model

$$Y_t = \prod_{k=1}^{K} Z_{tk}^{\delta_k} v_t \qquad t = 1, 2, \ldots$$

or

$$Y_t = \exp\{x_t'\beta + u_t\} \tag{2.16}$$

with

$$u_t = \rho u_{t-1} + \varepsilon_t \tag{2.17}$$

in which the ε_t are independently normally distributed with mean $-\frac{1}{2}\sigma_\varepsilon^2/(1 + \rho)$ and variance σ_ε^2.

Given a sample referring to period 1 to T we wish to estimate

$$\eta(x_{T+\theta}, u) = E[Y(x_{T+\theta})|u_1, \ldots, u_T] \tag{2.18}$$

This parameter can be rewritten as

$$\begin{aligned} \eta(x_{T+\theta}, u) &= E[\exp\{x_{T+\theta}'\beta + u_{T+\theta}\}|u_1, \ldots, u_T] \\ &= \exp\{x_{T+\theta}'\beta\}E[\exp\{e^{u_{T+\theta}}\}|u_1, \ldots, u_T] \\ &= \eta(x_{T+\theta})E[\exp\{\sum_{\tau=0}^{\theta-1} \rho^\tau \varepsilon_{T+\theta-\tau} + \rho^\theta u_T\}] \end{aligned} \tag{2.19}$$

with $\eta(x_{T+\theta}) = \exp\{x_{T+\theta}'\beta\}$.

If we assume that $u_T = u_T^0$ is the observed value of u at period T, we can write:

(2.20)
$$\eta(x_{T+\theta}, u) = \eta(x_{T+\theta}) \exp\{\rho^\theta u_T^0\} \exp\{\frac{-\frac{1}{2}\sigma_\varepsilon^2}{1-\rho^2}(\rho^{\theta-1} - \rho^{2\theta-2})\}$$
$$= \eta(x_{T+\theta}) \exp\{\rho^\theta u_T^0 - \tfrac{1}{2}\sigma_\varepsilon^2(\rho^{\theta-1} - \rho^{2\theta-2})/(1-\rho^2)\}$$

If we assume furthermore that ρ and σ_ε^2 are known, we have to estimate (apart from a known constant)

$$\eta(x_{T+\theta}) \, . \, \exp\{\rho^\theta u_T^0\}$$

For that purpose the following estimator is proposed:

(2.21)
$$\bar{H}_{T+\theta} = c \, . \, \exp\{x'_{T+\theta}\hat{\beta} + \rho^\theta \hat{u}_T\}$$

in which $\hat{\beta}$ and $\hat{u}_T$ are the generalized LS estimators of β and u_T^0:

(2.22)
$$\hat{\beta} = (X'P^{-1}X)^{-1}X'P^{-1}y$$

and

(2.23)
$$\hat{u}_T = e_T'Mu = (e_T - \ell_T)'u$$

e_T being the T-th unity vector. As can be seen from (2.21) the c is unspecified. As before we determine c by minimizing the MSE of $\bar{H}_{T+\theta}$. In order to derive the MSE of $E_{T+\theta}$ we write this estimator in a slightly different way:

$$\bar{H}_{T+\theta} = c \, . \, \eta(x_{T+\theta}) \, . \, \exp\{\ell'_{T+\theta}u + \rho^\theta(e_T - \ell_T)'u\}$$

or

(2.24)
$$\bar{H}_{T+\theta} = c\eta(x_{T+\theta}) \exp\{\ell'u\}$$

with

$$\ell = \ell_{T+\theta} + \rho^{\theta}(e_T - \ell_T) \tag{2.25}$$

It can easily be deduced that the relative MSE of $\bar{H}_{T+\theta}$ equals

$$\bar{\pi} = c^2 M_u(2\ell) - 2c\, M_u(\ell) + 1$$

and that the value of c for which this function reaches a minimum is

$$c_0 = \frac{M_u(\ell)}{M_u(2\ell)} \tag{2.26}$$

Substitution of (1.21) and (2.25) into (2.26) yields:

$$c_0 = \exp\{\tfrac{1}{2}\sigma^2(1 - 3\alpha_{T+\theta}) - \tfrac{3}{2}\sigma^2\rho^{2\theta}(1 - \alpha_T)\} \tag{2.27}$$

in which $\alpha_t = \ell_t'\ell_t = x_t'(X'P^{-1}X)^{-1}x_t$.

Hence the minimal MSE estimator of $\eta(x_{T+\theta}, u)$ (for known σ_ε^2 and ρ) becomes

$$\begin{aligned} \bar{Y}_{T+\theta} &= \exp\{-\tfrac{1}{2}\sigma_\varepsilon^2(\rho^{\theta-1} - \rho^{2\theta-2})/(1 - \rho^2)\}\, \bar{H}_{T+\theta} \\ &= \exp\{x_{T+\theta}'\hat{\beta} + \tfrac{1}{2}\sigma^2(1 - 3\alpha_{T+\theta}) + \\ &\quad + \rho^{\theta}[\hat{u}_T - \tfrac{3}{2}\sigma^2\rho^{\theta}(1 - \alpha_T) - \tfrac{1}{2}\sigma^2\rho^{-1}(1 - \rho^{\theta-1})]\} \end{aligned} \tag{2.28}$$

in which we substituted $\sigma_\varepsilon^2 = \sigma^2(1 - \rho^2)$.

It can be seen from (2.28) that $\bar{Y}_{T+\theta}$ consists of two parts the first one, $\exp\{x_{T+\theta}'\hat{\beta} + \frac{1}{2}\sigma^2(1 - 3\alpha_{T+\theta})\}$, can be considered as being predictor which results if we disregard the serial dependence of the errors and the second one, $\exp\{\rho^{\theta}[\hat{u}_T - \frac{3}{2}\sigma^2\rho^{\theta}(1 - \alpha_T) - \frac{1}{2}\sigma^2(1 - \rho^{\theta-1})/\rho]$, which improves the quality of the former estimator in the case of serial dependence. The importance of this correction clearly depends on the prediction period, if this period is far from the latest observation period, i.e. if θ is large, ρ^{θ} will tend to zero.

Having described the estimation procedure in the "ideal" situation where both σ^2 and ρ are known, we now consider the case where σ^2 is unknown and ρ is known.

Therefore, we write (2.28) as

$$(2.29)\quad \bar{Y}_{T+\theta} = \exp\{x'_{T+\theta}\hat{\beta} + \rho^{\theta}\hat{u}_T + \tfrac{1}{2}\sigma^2[1-3\alpha_{T+\theta}-3\rho^{2\theta}(1-\alpha_T) - \rho^{\theta-1}(1-\rho^{\theta-1})]\}$$

or

$$(2.30)\quad \bar{Y}_{T+\theta} = \exp\{x'_{T+\theta}\hat{\beta}+\rho^{\theta}\hat{u}_T + \xi\sigma^2\}$$

For unknown σ^2 the same estimation procedure is proposed as for the case $u \sim N[-\tfrac{1}{2}\sigma^2, \sigma^2 I]$, see Teekens and Koerts (1970, b):

$$(2.31)\quad \begin{cases} \bar{Y}_{T+\theta} = \exp\{x'_{T+\theta}\hat{\beta} + \rho^{\theta}\hat{u}_T\}g_{N-K}\left(\dfrac{N-K+1}{N-K}\,\xi S^2\right) & \text{for } \xi > 0 \\ \bar{Y}_{T+\theta} = \exp\{x'_{T+\theta}\hat{\beta} + \rho^{\theta}\hat{u}_T\} \,.\, \exp\{\xi S^2\} & \text{for } \xi \leq 0 \end{cases}$$

Finally, we have to deal with the case where both σ^2 and ρ are unknown. In this case we have to reconsider the β-estimator as well, since until now we used $\hat{\beta}$, which is a function of the unknown ρ. For this situation we propose that in (2.30) ρ be replaced by Durbin's $\hat{\rho}$ [2] and σ^2 be replaced by S^2. This proposal is based on the results of Rao and Griliches (1969). They compare several estimators of ρ on the bases of their (sumulated) M.S.E. and it turns out that "The Durbin $\hat{\rho}$ is significantly better[3] for high positive ρ, while at the same time not distinctly inferior to the other two methods for negative ρ's." Furthermore, it seems reasonable to drop the mixed approach (2.31) and simply replace σ^2 by S^2. The argument is that the conditions under which (2.31) has been derived are no longer valid if we deal with unknown ρ.

---

[2] See Durbin (1960.

[3] in MSE.

REFERENCES

Durbin, J. (1960). "The Fitting of Time-Series Models", Review of the International Statistical Institute, 28, pp. 233-243.

Johnston, J. (1963). Econometric Methods, McGraw-Hill Book Company Inc., N.Y.

Malinvaud, E. (1966). Statistical Methods of Econometrics, North Holland Publishing Company, Amsterdam.

Rao, P. and Z. Griliches (1969). "Small-Sample Properties of Several Two-Stage Regression Methods in the Context of Autocorrelated Errors", Journal of the American Statistical Association, 64, pp. 253-272.

Teekens, R. and J. Koerts (1970,a), Some Statistical Implications of the Log Transformation of Multiplicative Models, Report 7015 of the Econometric Institute, Netherlands School of Economics.

Teekens, R. and J. Koerts (1970, b), On a Minimal Mean Square Error Estimator of the Dependent Variable in Multiplicative Models Under Three Alternative Stochastic Assumptions, Report 7024 of the Econometric Institute, Netherlands School of Economics.