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A Class of Separability Flexible Functional Forms

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Quadratic flexible forms, such as the translog and generalized Leontief, are separability inflexible. That is, separability restrictions render them inflexible with regard to separable structures. A class of functional forms is proposed that is flexible with regard to general production structures and remains flexible regarding weakly separable structures when separability restrictions are imposed, thus permitting tests of the separability hypothesis. Additionally, the restricted forms are parsimonious; that is, they contain the mininum number of parameters with which flexibility can be achieved.

Key words: flexible functional forms, separability, weak separability.

Introduction

Of all the simplifying assumptions made in empirical production and demand analysis, perhaps the most powerful is separability. Separability of the production or utility function permits the researcher to focus on a subset of factor or consumer demands, thereby reducing the size of the problem to tractable dimensions.

One problem with the separability hypothesis, though, is that it is difficult to test. Blackorby, Primont, and Russell (BPR) have shown that restrictions necessary to impose weak separability on generalized quadratic functional forms (GQFFs), including the translog, the quadratic mean of order ρ (includes the quadratic and the generalized Leontief forms), and the generalized Cobb–Douglas, render these forms "inflexible" (the parameters of a flexibile form can be chosen so that the function value, marginal products, and elasticities can take on any arbitrary set of values at any point). BPR have shown that weak separability restrictions proposed by Berndt and Christensen (BC) overly restrict the form of the aggregators in the separable structure. For instance, in the case of the translog, BPR have shown that the BC weak separability test is actually a test for homothetic weak separability where all aggregator functions are Cobb–Douglas. The problem that arises with the BC tests for weak separability is that only special cases of weak separability are actually tested, and there is a danger of misinterpreting the results if the underlying function is characterized by a more general form of weak separability. On the basis of the BPR analysis, GQFFs are said to be separability inflexible.

Recently, there has been a resurgence of interest in the problem of testing separability. Pope and Hallam (PH) suggest exploiting duality relationships and employing production (profit) functions to test for weak separability in profit (production) functions. PH show that by using a quadratic production (profit) function, the BPR inflexibility objections can be circumvented when testing for separability in the profit (production) function. The PH approach, however, requires the assumption that producers are profit maximizers. Further, Lopez has shown that the quadratic profit function implies a quasi-homothetic production technology.

In this article, a method is proposed for generating separability-flexible functions so that parametric tests of separability may be undertaken. The approach is not limited to a specific functional form and permits testing separability within the primal framework.[1]

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No behavioral assumptions need to be made (although separability may be tested given the maintained hypothesis of cost minimization or profit maximization).[2] The suggested functional forms are similar in appearance to widely used GQFFs but contain some third and fourth order terms. Strictly speaking, the new forms cannot be considered Taylor series approximations because not all third and fourth order terms are included (all first and second order terms are included, however). Precisely which higher (than second) order terms are included in the specification depends on the type of separability being tested. The proposed models are all weak separability flexible by construction and, using results from Driscoll, McGuirk, and Alwang (DMA 1992a, b), are shown to be parsimonious (contain exactly the minimum number of parameters required for flexibility). While DMA (1992b) elaborate criteria for assessing flexibility and parsimony once separability is imposed, here, forms are designed that satisfy the criteria.

As a point of departure, the derivative conditions for separability are reviewed and are followed by a discussion of the BC restrictions developed for translog flexible forms. Using a couple of examples, the BC restrictions are shown to render the GQFF separability inflexible. Next, two methods for constructing parsimonious weak separability flexible forms are proposed. Although both methods generate parsimonious separability flexible restricted models, the corresponding unrestricted models are not parsimonious with respect to general production structures (they are flexible). The unrestricted models generated in the second approach are far more economical in their parameterizations than those obtained from the initial approach. Finally, a small Monte Carlo experiment is performed to assess the size and power of tests of weak separability based on the proposed forms.

Weak Separability and Some Inflexibility Results

Let $\boldsymbol{I}$ denote the set of indices of the input vector, $\boldsymbol{I} = \{1, 2, \ldots, n\}$. Partition I into m subsets, where $m \leq n$, and create a new set of indices $\boldsymbol{I}^*$, so that $\boldsymbol{I}^* = \{\boldsymbol{I}_1, \boldsymbol{I}_2, \ldots, \boldsymbol{I}_m\}$. The partition $\boldsymbol{I}^*$ defines a corresponding partition in the input vector $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m)$, where $\boldsymbol{x}_i$ is a subvector of the n-dimensional vector $\boldsymbol{x}$. The subvectors $\boldsymbol{x}_i$ each have z_i components.

The production function is said to be weakly separable in the partition $\boldsymbol{I}^*$ if it may be written

$$F(\boldsymbol{x}) = g(f_1(\boldsymbol{x}_1), f_2(\boldsymbol{x}_2), \ldots, f_m(\boldsymbol{x}_m)), \tag{1}$$

where g is strictly increasing and quasi-concave and each $f_i(\boldsymbol{x}_i)$ is strictly monotonic and quasi-concave. Weak separability of the rth group also may be expressed with the following familiar derivative condition that applies to ratios of marginal products of factors in a separable group:

$$\partial/\partial x_k \left| \frac{\partial F(\boldsymbol{x})/\partial x_i}{\partial F(\boldsymbol{x})/\partial x_j} \right| \equiv 0 \ \forall \ i, j \in \boldsymbol{I}_r, \ k \notin \boldsymbol{I}_r. \tag{2}$$

Weak Separability Inflexible GQFFs

Consider what the separability conditions imply for the following (second order) GQFF. A general quadratic flexible form can be written

$$F(\boldsymbol{x}) = \alpha_0 + \sum_{i \in \boldsymbol{I}} \alpha_i h_i(x_i) + \frac{1}{2} \sum_{i \in \boldsymbol{I}} \sum_{j \in \boldsymbol{I}} \beta_{ij} h_i(x_i) h_j(x_j), \tag{3}$$

where symmetry implies $\beta_{ij} = \beta_{ji}$. Using (2), if inputs i and j are to be separable from k, the following must hold:

$$\alpha_j \beta_{ik} - \alpha_i \beta_{jk} + \sum_{l \in \boldsymbol{I}} (\beta_{jl} \beta_{ik} - \beta_{il} \beta_{jk}) h_l(x_l) \equiv 0. \tag{4}$$

For the case where $h_l(x_l) = \ln(x_l)$ (the translog), BC suggest the following weak separability restrictions for (4) to hold globally [the form of the restrictions in (4) is actually independent of the form of h_l, as shown by BPR]:

(5a) $$\alpha_j\beta_{ik} - \alpha_i\beta_{jk} = 0 \quad \text{and}$$

(5b) $$\beta_{jl}\beta_{ik} - \beta_{il}\beta_{jk} = 0 \ \forall \ l \in \boldsymbol{I}.$$

BPR demonstrate that these restrictions require inputs i and j to be separable from all other inputs, not just those k inputs for which separability conditions are derived.

To illustrate how overly restrictive the conditions in (5) can be, consider the following two examples. First, the separability conditions developed for the technology

(6) $$F(\boldsymbol{x}) = g(f_1(x_1, x_2, x_3), f_2(x_4))$$

are indistinguishable from those derived for

(7) $$F(\boldsymbol{x}) = g\left(f_1\left(\sum_{i=1}^{3} \gamma_i h_i(x_i)\right), f_2(x_4)\right)$$

since each of the pairs (x_1, x_2), (x_2, x_3), and (x_1, x_3) are forced to be separable from all other inputs, including other inputs in $\boldsymbol{I}_1$.

From the discussion so far, it may appear that the separability restricted GQFF can provide a flexible approximation to the very simple technology

(8) $$F(\boldsymbol{x}) = g(f_1(x_1, x_2), f_2(x_3))$$

since there is only one input (x_3) from which the pair (x_1, x_2) can be separable. A second example demonstrates that a model imposing the BC restrictions on (8) is also inflexible. The restrictions, from (5), necessary to impose separability on (8) are

(9) $$\alpha_2\beta_{13} - \alpha_1\beta_{23} = 0,\ \beta_{12}\beta_{13} - \beta_{11}\beta_{23} = 0, \text{ and } \beta_{22}\beta_{13} - \beta_{12}\beta_{23} = 0.$$

Solving for α_2, β_{12}, and β_{22}, the GQFF reduces to a quadratic in two terms $(h_1(x_1) + \beta_{23}h_2(x_2)/\beta_{13})$ and $h_3(x_3)$. In other words, the separability restricted model from (8) is identical to that from

(10) $$F(\boldsymbol{x}) = g\left(f_1\left(\sum_{i=1}^{2} \gamma_i h_i(x_i)\right), f_2(x_3)\right).$$

Clearly, the specifications in (7) and (10) are too restrictive for purposes of testing for more general cases of weak separability.

Necessary Conditions for Flexibility of Weakly Separable Functions

Another way of identifying separability inflexible forms is to determine whether the form contains sufficient parameters to achieve flexibility. A functional form which is restricted to be separable but which contains fewer than the minimum number of parameters for flexibility is not separability flexible.

A functional form provides a flexible approximation to some underlying function if it is possible to choose its parameters in such a way that the function value, gradient, and hessian terms of the underlying function are *exactly* reproduced at an arbitrary point. Clearly this can be accomplished only if there are at least as many parameters as there are independent function value, gradient, and hessian effects. DMA (1992b) have shown that a weakly separable function has $1 + n + \Sigma_{i=1}^{m} z_i(z_i + 1)/2 + m(m - 1)/2$ independent effects, where z_i is the number of arguments in $\boldsymbol{x}_i$ of equation (1). By this rule, a flexible approximation to a weakly separable function must contain at least $1 + n + \Sigma_{i=1}^{m} z_i (z_i + 1)/2 + m(m - 1)/2$ parameters and therefore a separability flexible approximation to (8) requires at least nine parameters. Yet, for the case of (8), once the three parametric restrictions of the BC type are imposed, only seven parameters remain—too few to achieve flexibility.

Separability Flexible Functional Forms

Functional forms which are separability flexible can be developed easily by beginning with an explicit functional representation of the hypothesized separable structure (the restricted model). That is, a model is specified which is separable, flexible, and parsimonious by construction. A corresponding unrestricted form of the model is obtained by relaxing the separability restrictions. The unrestricted models suggested below are flexible for general, nonseparable structures; however, they are not parsimonious. This approach appears to be more promising than the current approach of starting with a model which is both flexible and parsimonious for general structures, imposing separability, and hoping that flexibility has not been lost in the process.

The separability restricted models suggested in this article are nonlinear in parameters (as are the BC models) but can be estimated using readily available econometric packages. See the "example" section below.

Structures without Any Trivial Aggregates

To begin, take a general representation of a weakly separable function with s aggregates (some of which may be trivial aggregates). Partition the n-vector $\boldsymbol{x}$ into $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_s$, and let z_i denote the number of elements in $\boldsymbol{x}_i$. Partition I in the same way. For generality, define

(11) $$\psi(Y) \equiv g(f_1(\boldsymbol{x}_1), f_2(\boldsymbol{x}_2), \ldots, f_s(\boldsymbol{x}_s)).$$

The aggregates f_{m+1} through f_s are trivial aggregates (functions of a single argument). Let $f_i(\boldsymbol{x}_i)\ \forall\ i = 1, s$ be represented by a GQFF (without an intercept term).

(12) $$f_i(\boldsymbol{x}_i) = \sum_{k \in \boldsymbol{I}_i} \alpha_{ik} h_k(x_k) + \frac{1}{2} \sum_{k \in \boldsymbol{I}_i} \sum_{l \in \boldsymbol{I}_i} \beta_{ikl} h_k(x_k) h_l(x_l).$$

Let the function $\psi(Y) = g(\boldsymbol{x})$ have the following representation:

(13) $$\psi(Y) = \alpha_0 + \sum_{i=1}^{s} f_i(\boldsymbol{x}_i) + \frac{1}{2} \sum_{\substack{i=1 \\ i \neq j}}^{s} \sum_{j=1}^{s} \beta_{ij} f_i(\boldsymbol{x}_i) f_j(\boldsymbol{x}_j).$$

Note that, except for the terms $f_i(\boldsymbol{x}_i)^2$, g is quadratic in $f_i(\boldsymbol{x}_i)$. The inclusion of the $f_i(\boldsymbol{x}_i)^2$ terms complicates estimation, admits the possibility of a complex solution to the system of equations described in (14) and (15) below, and is not necessary to achieve flexibility. They are omitted.

This model nests a GQFF and contains some (but not all) third and fourth order terms. Equation (13) represents a wide array of functional possibilities but includes some specifications that are familiar. For example, a model that resembles the translog is achieved if $\psi(Y) = \ln(Y)$ in (11) and $h_i(x_i) = \ln(x_i)\ \forall\ i$ in (12). The proposed functional form in (13) is weakly separable by construction and contains $1 + n + \Sigma_{i=1}^{s} z_i(z_i + 1)/2 + m(m - 1)/2$ parameters, a necessary condition for flexibility. It remains to be shown that (13) can portray the $1 + n + \Sigma_{i=1}^{s} z_i(z_i + 1)/2 + m(m - 1)/2$ *distinct* function value, gradient, and hessian effects of a weakly separable function at an arbitrary point (i.e., sufficiency).

If the flexibility of the specification in (13) is to be independent of the choices for $\psi(*)$ and $h_i(*)$, it is necessary to show first that for every independent gradient $(\partial Y/\partial x_i)$ and hessian $(\partial^2 Y/\partial x_i \partial x_j)$ term there is a corresponding independent gradient term of the form

$\partial\psi(Y)/\partial h_i(x_i)$ and hessian term of the form $\partial^2\psi(Y)/\partial h_i(x_i)\partial h_j(x_j)$. Sufficiency then can be established by demonstrating that the parameters of (13) can be chosen to solve the system of equations given by $\psi(Y)$, $\psi(Y)/\partial \boldsymbol{h}(\boldsymbol{x})$, and $\partial^2\psi(Y)/\partial \boldsymbol{h}(\boldsymbol{x})\partial \boldsymbol{h}(\boldsymbol{x})'$ at any point given arbitrary values of $\psi(Y)$, $\partial\psi(Y)/\partial \boldsymbol{h}(\boldsymbol{x})$, and $\partial^2\psi(Y)/\partial \boldsymbol{h}(\boldsymbol{x})\partial \boldsymbol{h}(\boldsymbol{x})'$.

By differentiating the identity in (11) with respect to x_j, the following identity is obtained:

$$\partial\psi(Y)/\partial h_j(x_j) \equiv \partial g(*)/\partial h_j(x_j) \equiv (\partial\psi(Y)/\partial Y)(\partial Y/\partial x_j)/(\partial h_j(x_j)/\partial x_j).$$

Once $h_j(x_j)$ and $\psi(Y)$ are specified, the derivatives $\partial h_j(x_j)/\partial x_j$ and $\partial\psi(Y)/\partial Y$ are known. This implies that there are as many independent gradient terms of form $\partial y/\partial x_j$ as there are independent gradient terms of the form $\partial\psi(Y)/\partial h_j(x_j)$.

Similarly, by finding the derivatives $\partial^2 Y/\partial x_j^2$ and $\partial^2 Y/\partial x_j\partial x_k$, a one-to-one correspondence between the independent terms of $\partial^2 Y/\partial\boldsymbol{x}\partial\boldsymbol{x}'$ and $\partial^2\psi(Y)/\partial \boldsymbol{h}(\boldsymbol{x})\partial \boldsymbol{h}(\boldsymbol{x})'$ can be established. Therefore, if (13) provides a flexible approximation at an arbitrary point to the independent function value $\psi(Y)$, gradients $\partial\psi(Y)/\partial \boldsymbol{h}(\boldsymbol{x})$, and hessian terms $\partial^2\psi(Y)/\partial \boldsymbol{h}(\boldsymbol{x})\partial \boldsymbol{h}(\boldsymbol{x})'$, it also provides a flexible approximation at an arbitrary point to the independent Y, gradients $\partial Y/\partial\boldsymbol{x}$, and hessian terms $\partial^2 Y/\partial\boldsymbol{x}\partial\boldsymbol{x}'$.

The system of equations given by $\psi(Y)$, $\partial\psi(Y)/\partial \boldsymbol{h}(\boldsymbol{x})$, and $\partial^2\psi(Y)/\partial \boldsymbol{h}(\boldsymbol{x})\partial \boldsymbol{h}(\boldsymbol{x})'$ is now shown to have a solution at any arbitrary point and for any arbitrary values of $\psi(Y)$, $\partial\psi(Y)/\partial \boldsymbol{h}(\boldsymbol{x})$, and $\partial^2\psi(Y)/\partial \boldsymbol{h}(\boldsymbol{x})\partial \boldsymbol{h}(\boldsymbol{x})'$. The gradient terms obtained from (13) are

$$\text{(14)}\quad \partial\psi(Y)/\partial h_k(x_k) = \left(1 + \sum_{j\neq i} \beta_{ij} f_j(\boldsymbol{x}_j)\right)\left(\alpha_{ik} + \sum_{l\in\boldsymbol{I}_i} \beta_{ikl} h_l(x_l)\right), \quad k\in\boldsymbol{I}_i, \quad i = 1, s,$$

and the hessian terms are

$$\text{(15a)}\quad \partial^2\psi(Y)/\partial h_k(x_k)\partial h_l(x_l) = \beta_{ikl}\left(1 + \sum_{j=1}^{s} \beta_{ij} f_j(\boldsymbol{x}_j)\right), \quad k, l\in\boldsymbol{I}_i, \quad i = 1, s,$$

and

$$\text{(15b)}\quad \partial^2\psi(Y)/\partial h_k(x_k)\partial h_l(x_l) = \beta_{ij}\left(\alpha_{ik} + \sum_{r\in\boldsymbol{I}_i} \beta_{ikr} h_r(x_r)\right)\left(\alpha_{il} + \sum_{r\in\boldsymbol{I}_j} \beta_{ilr} h_r(x_r)\right) \forall k\in\boldsymbol{I}_i, l\in\boldsymbol{I}_j.$$

DMA (1992b) show that for every (i, j) pair where $i \neq j$, there is only one independent hessian term of the form (15b). Whatever the values of the other parameters, β_{ij} can be chosen to solve the single independent hessian term in each (i, j) pair. Notice that there are $m(m - 1)/2$ β_{ij} parameters and $m(m - 1)/2$ independent hessian terms of the form (15b). Similarly, choose β_{ikl} to solve (15a) given values for the β_{ij}s. Equation (12) contains $z_i(z_i + 1)/2$ β_{ikl} parameters for each partition, one for each independent hessian element in (15a).

Continuing, the parameters α_{ik} can now be chosen to satisfy the gradients in (14) given values of the other parameters. Again, equation (12) contains one α_{ik} for every gradient term. Finally, α_0 of equation (13) may be chosen to solve $\psi(Y)$. Since the system of equations $\psi(Y)$, $\partial\psi(Y)/\partial \boldsymbol{h}(\boldsymbol{x})$, and $\partial^2\psi(Y)/\partial \boldsymbol{h}(\boldsymbol{x})\partial \boldsymbol{h}(\boldsymbol{x})'$ has a solution, the specification in (13) is weak separability flexible.

An unrestricted model corresponding to (13) is developed by relaxing the parametric restrictions in (13). The following model is obtained:

$$\text{(16)}\quad \psi(Y) = \alpha_0 + \sum_{i=1}^{n} \alpha_i h_i(x_i) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \beta_{ij} h_i(x_i) h_j(x_j) + \sum_{\substack{i=1\\ i\neq j}}^{s}\sum_{j=1}^{s}\sum_{k\in\boldsymbol{I}_i}\sum_{l\in\boldsymbol{I}_i}\sum_{q\in\boldsymbol{I}_j} \beta_{ijklq} h_k(x_k) h_l(x_l) h_q(x_q)$$

$$+ \sum_{\substack{i=1 \\ i \neq j}}^{s} \sum_{j=1}^{s} \sum_{q \in I_j} \sum_{r \in I_j} \sum_{k \in I_i} \beta_{ijqrk} h_q(x_q) h_r(x_r) h_k(x_k)$$

$$+ \sum_{\substack{i=1 \\ i \neq j}}^{s} \sum_{j=1}^{s} \sum_{q \in I_j} \sum_{r \in I_j} \sum_{k \in I_i} \sum_{l \in I_i} \beta_{ijqrkl} h_q(x_q) h_r(x_r) h_k(x_k) h_l(x_l).$$

Equation (16) is flexible with regard to general production structures since it contains a second order GQFF as well as other higher order terms. It is, however, very cumbersome; the unrestricted model corresponding to the separable model constructed for $g(f_1(x_1, x_2), f_2(x_3))$ contains 18 parameters and for $g(f_1(x_1, x_2), f_2(x_3), f_3(x_4))$ contains 34 parameters.

Structures with Trivial Aggregates

Recall that the vector $\boldsymbol{x}$ is ordered and partitioned so that aggregates $f_{m+1}(\boldsymbol{x}_{m+1})$ through $f_s(\boldsymbol{x}_s)$ are trivial aggregates ($\boldsymbol{x}_{m+1}$ through $\boldsymbol{x}_s$ each have a single element). *Large* economies in the parameterization of the unrestricted model can be achieved by treating these trivial aggregates in a different fashion. Specifically, alter (13) so that

$$(17) \qquad \psi(Y) = \alpha_0 + \sum_{i=1}^{s} f_i(\boldsymbol{x}_i) + \sum_{\substack{i=1 \\ i \neq j}}^{m} \sum_{j=1}^{m} \beta_{ij} f_i(\boldsymbol{x}_i) f_j(\boldsymbol{x}_j)$$

$$+ \sum_{i=1}^{m} \sum_{j=m+1}^{s} \beta_{ij} f_i(\boldsymbol{x}_i) h_j(x_j) + \sum_{\substack{i=m+1 \\ i \neq j}}^{s} \sum_{j=m+1}^{s} \beta_{ij} h_i(x_i) h_j(x_j).$$

The model in (17) is separability flexible as well as parsimonious. The proof of sufficiency is entirely analogous to the proof given for (13). The specification in (17) has at least two advantages over that in (13). First, the restricted model, although it contains an identical number of parameters, contains fewer terms and fewer restrictions, and is therefore easier to estimate. Second, the unrestricted model corresponding to (17),

$$(18) \qquad \psi(Y) = \alpha_0 + \sum_{i=1}^{n} \alpha_i h_i(x_i) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} h_i(x_i) h_j(x_j)$$

$$+ \frac{1}{2} \sum_{i=1}^{m} \sum_{j=m+1}^{s} \sum_{k \in I_i} \sum_{l \in I_i} \beta_{ijkl} h_j(x_j) h_k(x_k) h_l(x_l)$$

$$+ \sum_{\substack{i=1 \\ i \neq j}}^{m} \sum_{j=1}^{m} \sum_{k \in I_i} \sum_{l \in I_i} \sum_{q \in I_j} \beta_{ijklq} h_k(x_k) h_l(x_l) h_q(x_q)$$

$$+ \sum_{\substack{i=1 \\ i \neq j}}^{m} \sum_{j=1}^{m} \sum_{q \in I_j} \sum_{r \in I_j} \sum_{k \in I_i} \beta_{ijqrk} h_q(x_q) h_r(x_r) h_k(x_k)$$

$$+ \sum_{\substack{i=1 \\ i \neq j}}^{m} \sum_{j=1}^{m} \sum_{q \in I_j} \sum_{r \in I_j} \sum_{k \in I_i} \sum_{l \in I_i} \beta_{ijqrkl} h_q(x_q) h_r(x_r) h_k(x_k) h_l(x_l),$$

maintains flexibility with regard to general production structures but contains fewer parameters and fewer higher (than second) order terms than the specification in (16). *When there is only one nontrivial aggregate (a common practical case), the last three summations* *in (18) disappear and no fourth order terms appear in the unrestricted model.* As a result, the unrestricted model corresponding to $g(f_1(x_1, x_2), f_2(x_3))$ contains only 13 parameters and the unrestricted model corresponding to $g(f_1(x_1, x_2), f_2(x_3), f_3(x_4))$ contains 21 parameters (a second order quadratic form in four arguments contains 15 parameters).

Monte Carlo Experiment

To obtain some notion of the size and power of tests of weak separability based on the proposed forms, the following Monte Carlo experiment is performed. First, we construct a separability flexible form capable of testing for the following three-input structure:

$$F(\mathbf{x}) = g(f_1(x_1, x_2), f_2(x_3)). \tag{19}$$

Because there is one trivial aggregate, the (separability flexible) restricted model and corresponding unrestricted model (flexible with regard to general production structures) are constructed using equations (17) and (18), where $\psi(Y) = \ln(Y)$ and $h_i(x_i) = \ln(x_i)$. Specifically, the restricted model is

$$\begin{aligned}\ln(Y) = \alpha_0 &+ \alpha_{11}\ln(x_1) + \alpha_{12}\ln(x_2) + \alpha_{21}\ln(x_3) + \tfrac{1}{2}\beta_{111}\ln(x_1)^2 \\ &+ \tfrac{1}{2}\beta_{122}\ln(x_2)^2 + \beta_{112}\ln(x_1)\ln(x_2) + \tfrac{1}{2}\beta_{211}\ln(x_3)^2 \\ &+ \beta_{12}\ln(x_3)(\alpha_{11}\ln(x_1) + \alpha_{12}\ln(x_2) + \tfrac{1}{2}\beta_{111}\ln(x_1)^2 \\ &\qquad + \tfrac{1}{2}\beta_{122}\ln(x_2)^2 + \beta_{112}\ln(x_1)\ln(x_2)).\end{aligned} \tag{20}$$

If an assumption of cost minimizing producer behavior is maintained, the production function in (20) may be supplemented by a set of first order conditions when testing for separability. The first order conditions, $p_i = \lambda f_i$, imply that $M_i = \partial\ln(y)/\partial\ln(x_i)/(\Sigma_i\partial\ln(y)/\partial\ln(x_i))$, where M_i is a cost share (see Christensen, Jorgensen, and Lau). For the case at hand, these first order conditions are

$$M_i = K_i/(K_1 + K_2 + K_3) + \epsilon_i \ \forall \ i = 1, 2, \tag{21}$$

where

$$K_1 = (\alpha_{11} + \beta_{111}\ln(x_1) + \beta_{112}\ln(x_2))(1.0 + \beta_{12}\ln(x_3)),$$

$$K_2 = (\alpha_{12} + \beta_{112}\ln(x_1) + \beta_{122}\ln(x_2))(1.0 + \beta_{12}\ln(x_3)), \quad \text{and}$$

$$\begin{aligned}K_3 = \alpha_{21} + \beta_{211}\ln(x_3) + \beta_{12}(\alpha_{11}\ln(x_1) + \alpha_{12}\ln(x_2) + \tfrac{1}{2}\beta_{111}\ln(x_1)^2 \\ + \tfrac{1}{2}\beta_{122}\ln(x_2)^2 + \beta_{112}\ln(x_1)\ln(x_2)).\end{aligned}$$

To determine the form of the unrestricted model, expand the last term in (20) to see what cross-product terms are involved. The unrestricted model contains all of the terms in (20) but there are no parametric restrictions imposed. The unrestricted model is

$$\begin{aligned}\ln(Y) = \alpha_0 + \sum_{i=1}^{3}\alpha_i\ln(x_i) + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\beta_{ij}\ln(x_i)\ln(x_j) + \tfrac{1}{2}\gamma_{113}\ln(x_1)^2\ln(x_3) \\ + \tfrac{1}{2}\gamma_{223}\ln(x_2)^2\ln(x_3) + \gamma_{123}\ln(x_1)\ln(x_2)\ln(x_3).\end{aligned} \tag{22}$$

It can be verified that all terms in equation (22) are represented by the first two lines of equation (18); that is, since there is a single nontrivial aggregate, the last three lines of equation (18) are irrelevant.

Assuming cost minimizing behavior, the first order conditions are again

$$M_i = K_i/(K_1 + K_2 + K_3) + \epsilon_i \ \forall \ i = 1, 2, \tag{23}$$

where the K_i are now

$$K_1 = \alpha_1 + \beta_{11}\ln(x_1) + \beta_{12}\ln(x_2) + \beta_{13}\ln(x_3) + \gamma_{113}\ln(x_1)\ln(x_3) + \gamma_{123}\ln(x_2)\ln(x_3),$$

$$K_2 = \alpha_2 + \beta_{12}\ln(x_1) + \beta_{22}\ln(x_2) + \beta_{23}\ln(x_3) + \gamma_{223}\ln(x_2)\ln(x_3) + \gamma_{123}\ln(x_1)\ln(x_3),$$

and

$$K_3 = \alpha_3 + \beta_{13}\ln(x_1) + \beta_{23}\ln(x_2) + \beta_{33}\ln(x_3) + \tfrac{1}{2}\gamma_{113}\ln(x_1)^2 + \tfrac{1}{2}\gamma_{223}\ln(x_2)^2 + \gamma_{123}\ln(x_1)\ln(x_2).$$

Both the unrestricted and restricted models are nonlinear in parameters; therefore, it is convenient to employ the Gallant–Jorgensen chi-square statistic to test the separability hypothesis. Briefly, the test is conducted by first estimating the unrestricted model via an ITSUR procedure, saving the residual covariance matrix. The coefficients of the restricted model are then estimated via a SUR procedure using the residual covariance matrix of the unrestricted model. The test statistic is $T_0 = nS^{**} - nS^{*}$. Define $S(\Theta) = q'(\Theta)(\Sigma^{-1} \otimes I)q(\Theta)/n$. S^{**} and S^{*} are $S(\Theta)$ evaluated for the restricted and unrestricted models, respectively; $q(\Theta)$ is the stacked error vector from the model evaluated at the converged parameter values, Θ; and Σ is the error covariance matrix. The statistic is distributed χ^2_k, where k is the number of restrictions.

The Monte Carlo experiment proceeds as follows. First, 50 observations of three inputs are generated from a multivariate log-normal distribution:

$$\ln(X) \sim N\begin{bmatrix} 0.239 \\ 0.582 \\ 0.391 \end{bmatrix}, \quad \begin{bmatrix} 0.026 & & \\ 0.039 & 0.066 & \\ 0.041 & 0.063 & 0.065 \end{bmatrix}.$$

Input data are fairly collinear. Next, output is calculated from a production function having the general form of a fourth order translog,

$$\ln(y_t) = \alpha_0 + \sum_{i=1}^{3} \alpha_i \ln(x_{it}) + \sum_{i=1}^{3}\sum_{j>i} \beta_{ij}\ln(x_{it})\ln(x_{jt}) + \sum_{i=1}^{3}\sum_{j>i}\sum_{k>j} \delta_{ijk}\ln(x_{it})\ln(x_{jt})\ln(x_{kt}) + \sum_{i=1}^{3}\sum_{j>i}\sum_{k>j}\sum_{l>k} \gamma_{ijkl}\ln(x_{it})\ln(x_{jt})\ln(x_{kt})\ln(x_{lt}), \tag{24}$$

where x_{it} are inputs and y_t is output. All technologies used in the experiment are generated by selecting parameter values for equation (24) (see below and table 1 for details). A set of price data consistent with profit maximization and cost minimization subject to an output constraint is generated as $p_i = p_y f_i$, where the f_i are marginal products from equation (24) and output price, p_y, is set to unity for all observations. Multiplicative disturbances (2% standard deviation) are added to output; input expenditures receive an additive disturbance (5–7% standard deviation). As is customary, output and input data are scaled by their geometric mean.

Once the data are generated, the hypothesis that inputs 1 and 2 are weakly separable from input 3 (12-3) is tested using two approaches. First, only the production function is employed. That is, the unrestricted model consists only of equation (22) and the restricted model consists only of (20). No hypotheses containing producer behavior are maintained. Second, the 12-3 weak separability hypothesis is tested while maintaining cost minimizing producer behavior. The unrestricted model consists of equations (22)–(23) and the restricted model consists of equations (20)–(21).

One hundred replications of the experiment are performed for each of five technologies having the general fourth order translog form in equation (24). Values of the parameters

Table 1. Parameters of Translog Technologies 1–5

	α_0	α_1	α_2	α_3	β_{11}	β_{12}	β_{13}	β_{22}	β_{23}	β_{33}
TECH 1	−.367	.323	.366	.339	−.145	.205	.072	−.242	.203	−.224
TECH 2	−.367	.323	.366	.339	−.145	.205	.072	−.242	.203	−.224
TECH 3	−.367	.323	.366	.339	−.045	−.101	.046	−.057	.053	−.224
TECH 4	−.367	.199	.201	.540	−.043	.048	.017	−.033	.018	−.051
TECH 5	−.367	.199	.201	.540	−.043	.048	.017	−.033	.018	−.051

of equation (24) for each of the technologies are given in table 1. Technology 1 (*TECH* 1) is a nonseparable second order translog, technology 2 (*TECH* 2) is a nonseparable translog containing some third order terms, technology 3 (*TECH* 3) is a homothetically separable second order translog (with Berndt–Christensen restrictions imposed), technology 4 (*TECH* 4) is a weakly separable translog that contains some third order terms [generated with equations (12) and (17)], and technology 5 (*TECH* 5) is a weakly separable quadratic of quadratics form.[3]

The results of the experiment are recorded in table 2. Ideally, separability always will be rejected for technologies 1 and 2. The system test performs fairly well, correctly rejecting separability in 85% and 91% of replications for technologies 1 and 2, respectively. The single equation test is found to have poor power in the experiments, failing to reject false nulls in 93–94% of the replications for technologies 1 and 2. However, preliminary investigation reveals that the power of the single equation test improves somewhat when output disturbances are reduced to .1%.[4] At the bottom of table 2, results for the reduced error scenario are reported. The single equation test correctly rejects separability in 21% and 72% of replications of technologies 1 and 2, respectively. The weak power (especially with regard to technology 1) may indicate that the separability flexible form is sufficiently flexible for modeling somewhat more general production structures.

For technologies 3, 4, and 5, separability should be rejected in 5% of all replications; that is, the nominal size of the test is .05. The actual size of the system test is larger than expected and the true nulls are rejected in 16%, 15%, and 19% of replications for technologies 3, 4, and 5, respectively. When output disturbances have a 2% standard deviation, the actual size of the single equation test is very close to the nominal size of the test. When output disturbances are reduced to .1% standard deviation, the nominal size of the test increases modestly to about .15 (see results for technologies 3, 4, and 5 at the bottom of table 2).

Summary

Separability flexible forms have been developed that permit testing a weak separability hypothesis without imposing any unwanted structure on the restricted model. The models are closely related to the GQFFs in common use but involve some additional third and, when there are two or more nontrivial aggregates, fourth order terms. The restricted models are all parsimonious and their corresponding unrestricted models are flexible with regard to general production or utility structures. Apparently, the only cost of achieving flexibility in both the restricted and unrestricted models is that the unrestricted models are not parsimonious. When testing for separable structures characterized by a single nontrivial aggregate, specifications can be developed that contain far fewer parameters than a fully parameterized third order approximation. Therefore, data requirements and computational burdens may be less excessive than previously thought.

Some preliminary Monte Carlo evidence suggests that the system tests of weak separability have reasonable size and power. The power of the single equation test is poor even when data are measured with great accuracy. Taken together, these results indicate

Table 1. Continued

δ_{111}	δ_{112}	δ_{113}	δ_{221}	δ_{222}	δ_{223}	δ_{123}	γ_{1111}	γ_{1112}	γ_{1122}	γ_{2221}	γ_{2222}
.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0
.0	.0	−.026	.0	.0	−.011	.073	.0	.0	.0	.0	.0
.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0
.0	.0	−.004	.0	.0	−.003	.004	.0	.0	.0	.0	.0
−.022	−.026	−.004	−.001	−.022	−.003	.004	−.0005	.001	−.0012	.0008	−.0003

Table 2. Monte Carlo Results for Separability Tests

Output Disturbances with 2% Standard Deviations				
	Rejection (%)	χ^2 Mean	χ^2 Variance	χ^2 Value*
Nonseparable Technologies:				
TECH 1				
System	85	45.95	956.98	N/A
Single Equation	7	4.02	9.76	N/A
TECH 2				
System	91	52.31	1111.01	N/A
Single Equation	6	3.93	8.46	N/A
Separable Technologies:				
TECH 3				
System	16	13.19	206.87	35.45
Single Equation	7	4.04	10.44	9.75
TECH 4				
System	15	11.55	132.63	29.93
Single Equation	5	4.30	9.35	8.98
TECH 5				
System	19	15.68	477.68	37.68
Single Equation	5	3.65	8.15	8.64

Output Disturbances with .1% Standard Deviations (Single Equation Tests Only)				
	Rejection (%)	χ^2 Mean	χ^2 Variance	χ^2 Value*
Nonseparable Technologies:				
TECH 1	21	6.45	19.13	N/A
TECH 2	72	17.57	107.47	N/A
Separable Technologies:				
TECH 3	13	5.36	13.98	11.25
TECH 4	16	5.96	15.16	12.94
TECH 5	16	5.68	12.56	11.31

* Minimum χ^2 values for which the actual rejection rate equals the nominal rate of 5%. The nominal critical values ($\alpha = .05$) for χ^2_{12} (system test) and χ^2_4 (single equation test) are 21 and 9.47, respectively.

that powerful parametric tests of weak separability may, after all, require some assumption about producer behavior.

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Notes

[1] In order to justify modeling a subset of factor demands as a function only of those factor prices and expenditures on the subset, it is separability of the *primal* function that must be established.

[2] This is true only in production analyses. In demand analyses, one must assume that consumers maximize utility subject to a budget constraint in order to derive a set of first order conditions. A system of equations implied by these first order conditions can be estimated, whereas the utility function cannot be estimated.

[3] Specifically, $\ln(Y)$ is quadratic in $\ln(z_1)$ and $\ln(z_2)$; $\ln(z_1)$ is quadratic in $\ln(x_1)$ and $\ln(x_2)$, while $\ln(z_2) = \ln(x_3)$.

[4] When data disturbances are so small, the ITSUR procedure often will not converge. The unrestricted models tend to fit very well and at some iteration, Σ, the error covariance matrix, cannot be inverted. For the single equation tests, a nonlinear OLS procedure and a likelihood ratio test were substituted. The experiments could not be repeated for the system tests which must make use of the ITSUR estimator.

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