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ON THE COMPETITIVE FIRM UNDER PRODUCTION UNCERTAINTY*

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Risky production functions which are commonly in use are shown to be very restrictive. In particular, such functions cannot describe technologies where inputs marginally reduce risk. A simple production function which avoids these restrictions is posited and alternative estimation procedures are discussed. Both maximum likelihood and multistage estimators are discussed.

Recently, a number of authors have discussed comparative static results under price uncertainty for the single-product firm (Sandmo [18], Leland [14], and Blair [2]). In agriculture, however, production uncertainty must also be considered an important aspect of the firm's environment. A number of authors have investigated some special cases of technological or production risk in various contexts (e.g., Stiglitz [19]. Feldstein [7], and Rothenberg and Smith [16]). This paper attempts to investigate the effects of production uncertainty on factor use under a more general (and plausible) case of stochastic technology. Results indicate that risk aversion does not necessarily imply less factor usage as compared to the risk-neutral firm. Furthermore, characteristics of inputs will be established for which input use is generally greater than (less than) the risk-neutral case. It is further argued that these results may be anticipated from mean-variance utility analysis. As a by-product of the investigation, insight is gained concerning more flexible functional forms for theoretical and empirical investigations of economic behavior under technological risk.

Supposedly, the competitive firm is faced by three major sources of uncertainty: output price uncertainty, input price uncertainty, and production response uncertainty. The effects of random output prices have been examined by Sandmo, and the effects of random input prices by Blair. The indication of Sandmo's work is that the risk-averse firm chooses an optimal output, q^*, such that $E(P) > C'(q^*)$ where $C'(.)$ denotes marginal cost as a function of output and $E(P)$ denotes expected price[1]. In applying this result to factor demands, an assumption of complementarity among inputs is sufficient to ensure that all inputs move in the same direction as output—i.e., factor use will be diminished as a result of risk aversion (Ferguson [8]). Assuming random input prices, on the other hand, Blair found that the risk-averse firm produces with $MRP_i - E(MFC_i) > 0$, where MRP_i is marginal revenue

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[1] It is assumed that $C'(.)$ is positive.

product of factor i and MFC_i is marginal factor cost. The case of technological uncertainty, however, has not been examined on an equally general basis.

To examine technological uncertainty, let profits be denoted by

$$\pi = Pq - \sum_{j=1}^{N} \gamma_j X_j$$

where P is the fixed price of output, q is output, γ_j are fixed factor prices, and X_j are factor quantities. Let $q = F(X_1 \ldots X_N, \varepsilon)$ denote random technology where ε is a random disturbance. Finally, let U be the producer's risk-averse utility function defined over profits. First-order conditions for expected utility maximization are (second-order conditions are assumed to hold)[2]

$$E[U'(\eta)(PF_j - \gamma_j)] = 0, \qquad j = 1 \ldots N \tag{1}$$

where $U'(\pi) > 0$ is the marginal utility of income, and F_j $(= \partial q/\partial X_j)$ is the random marginal product. Rewriting (1) by adding and subtracting $E[U'(\pi)\, PE(F_j)]$ yields

$$E\{U'(\pi)\,[PF_j - PE(F_j)]\} + [PE(F_j) - \gamma_j]E[U'(\pi)] = 0, \quad j = 1 \ldots N \tag{2}$$

where the first term in (2) can be written as

$$PE\{\{U'(\pi) - U'[E(\pi)]\}\,[F_j - E(F_j)]\} \tag{3}$$

by adding and subtracting $E\{U'[E(\pi)]\, F_j\}$ and noting that $U'[E(\pi)]$ is nonrandom.

One of the most popular theoretical and empirical production functions is of the form

$$q = F(X) = f(X)g(\varepsilon) \tag{4}$$

(see Stiglitz [19], Walters [20]; Rothenberg and Smith [16]; Blair and Lusky [3]; and Zellner, Kmenta, and Dreze [22]). For this form, setting $F_j = \partial q/\partial X_j$ and $E[g(\varepsilon)] = \mu$, one obtains

$$[F_j - E(F_j)]\,[q - E(q)] = [g(\varepsilon) - \mu]^2 f_j f. \tag{5}$$

Clearly, for f_j, $f > 0$, equation (5) implies that $q > E(q)$ whenever $F_j > E(F_j)$ and, hence, $\pi > E(\pi)$ and $U'(\pi) < U'[E(\pi)]$ (since $U'' < 0$). Conversely, $q < E(q)$ whenever $F_j < E(F_j)$ and, hence, $\pi < E(\pi)$ and $U'(\pi) > U'[E(\pi)]$. Thus, (3) is negative and since $E[U'(\pi)] > 0$, one finds, using (2), that

$$PE(F_j) > \gamma_j \tag{6}$$

[2] Second-order conditions dictate that

$$\sum_i \sum_j dX_j\, dX_i \frac{\partial E(U)}{\partial X_i\, \partial X_j}$$

be negative or

$$\sum_j \sum_i E\left[U'(\pi)\frac{\partial^2 \pi}{\partial X_i\, \partial X_j} + U''\frac{\partial \pi}{\partial X_i}\frac{\partial \pi}{\partial X_j}\right] dX_i\, dX_j < 0.$$

at the optimum.[3] In other words, all factor uses are less under risk aversion than under risk neutrality.[4]

It is often argued, however, that risk can have the effect of stimulating factor use (Carlson [4], Just and Pope [12]). For example, the risk associated with machine failure may cause the firm to acquire capital to be held in reserve for such failures. Additional labor may also be acquired because of either random availability or random transmission of labor services to output. There are a number of similar inputs in agriculture. For example, use of pesticides, irrigation, frost protection, and disease-resistant seed varieties, and certain forms of capital all tend to reduce risk—at least in some cases. But for these cases, it is possible that inputs are used in greater quantities under risk aversion than under risk neutrality. Unfortunately, the commonly used multiplicative specification in (4) does not admit the possibility of decreasing risk. Indeed, (4) implies that changes in variance are positively related to factor changes, i.e.,

$$[\partial V(q)/\partial X_j] = 2f(X)f_j(X)V[g(\varepsilon)] > 0$$

under the usual assumptions ($f, f_j > 0$). But this property has a direct correspondence with the arguments made above, since from (5)

$$E[q - E(q)][F_j - E(F_j)] = \text{Cov}(q, F_j) = (1/2)[\partial V(q)/\partial X_j].$$

That is, if for all states of nature $q > E(q)$ whenever $F_j > E(F_j)$, then $\text{Cov}(q, F_j)$ and, thus, $\partial V(q)/\partial X_j$ are always positive (and conversely).

A number of functional forms are capable of admitting decreasing risk and its associated response but one simple form possessing the needed generality may be represented as

$$q = F(X) = f(X) + h(X)\varepsilon, \qquad E(\varepsilon) = 0. \tag{7}$$

In this case, $q - E(q) = h(X)\varepsilon$ and $F_j - E(F_j) = h_j\varepsilon$. Therefore, if $h > 0$ and $h_j < 0$, then $\varepsilon > E(\varepsilon)$ implies that $q > E(q)$, $F_j < E(F_j)$, and $\pi > E(\pi)$; hence, $U'(\pi) < U'[E(\pi)]$. Conversely, if $\varepsilon < 0$ then $F_j > E(F_j)$, $q < E(q)$, and $\pi < E(\pi)$; hence, $U'(\pi) > U'[E(\pi)]$. But it is immediately evident in this case that (3) is positive and, since $E[U'(\pi)] > 0$, optimal use for factor j satisfies

$$P\, E(F_j) < \gamma_j.$$

Thus, factor use under risk aversion exceeds risk-neutral factor application for those factors with $h > 0$, $h_j < 0$ (this also holds for $h < 0$, $h_j > 0$).

[3] This same property can also be shown for the random coefficient model investigated under expected profit maximization by Feldstein. For example, consider the simple case with

$$q = F(X) \equiv f(X)e^{\varepsilon X}, \qquad \varepsilon \sim N(0, \sigma)$$

(e.g., $q = z^{\alpha+\varepsilon} = z^{\alpha}\, e^{\varepsilon \ln z}$ where $X = \ln z$). In this case

$$F_j - E(F_j) = f'(X)(e^{\varepsilon X} - e^{\sigma x^2/2}) + \frac{f(X)}{Z}(\varepsilon e^{\varepsilon X} - \sigma X)$$

$$q - E(q) = f(X)(e^{\varepsilon X} - e^{\sigma x^2/2}).$$

Clearly, both $F_j - E(F_j)$ and $q - E(q)$ have expectation zero and are increasing in ε. Thus, following Blair's method, the expectation of (5) is a covariance and must be positive.

[4] One can also make inferences about the effects of increasing risk by making stronger assumptions than have been made here concerning the concavity of $\partial U/\partial X_j$ in ε (Rothschild and Stiglitz).

Although it may seem odd to one accustomed to usual stochastic specifications that marginal physical productivity may be negatively correlated with output, this is indeed plausible. Consider an agricultural example where ε is a measure of the goodness of weather. It is reasonable to suppose that a 'high' value of the state of nature implies large output. However, at the margin, the productivity of frost protection and irrigation variables are probably quite low when optimum weather conditions are realized (where ε and thus q are large. Conversely, the marginal productivity of irrigation would be great in a situation of drought, and the marginal productivity of frost protection would be great when damaging frosts occur (where ε and thus q are small). Such a relationship between output and marginal productivity would imply that the risk-averse firm purchases relatively more of these factors—a result which seems to be in accordance with observed behaviour but inconsistent with theoretical specifications such as (4).

Empirical Possibilities

Since the general functional form in (7) leads to more plausible theoretical results than some of the popular and widely used functions, it is also interesting to consider empirical possibilities. For example, the Cobb-Douglas function with log-linear disturbances,

$$q = AX^{\alpha}e^{\varepsilon}, \qquad E(\varepsilon) = 0, \tag{8}$$

is perhaps the most widely used empirical production function. However, letting

$$\begin{aligned} f(X) &= AX^{\alpha} \\ g(\varepsilon) &= e^{\varepsilon} \end{aligned} \tag{9}$$

one finds that equation (8) is clearly a special case of equation (4) and, hence, all of the criticism relating to that popular theoretical formulation (noted above) is applicable. This is also clearly true for any empirical production function with a multiplicative disturbance of the form in (9); hence, the criticism developed above applies also for the usual stochastic specification of the translog [5], and generalized power production functions (which includes transcendental functions, see [6]) as well. It thus seems that empirical use of the general specification in (7) is also desirable.

Unfortunately, however, empirical use of (7) is somewhat more cumbersome. For example, if f in (7) follows any of the conventional neoclassical log linear forms (e.g., Cobb-Douglas, translog, transcendental, etc.), then estimation by linear methods is no longer possible except in the special case where $h \equiv f$. That is, one can rewrite (7) as

$$q = f(X)\left[1 + \frac{h(X)}{f(X)}\varepsilon\right]$$

or

$$\ln q = (\ln x)'\alpha + \varepsilon^* \tag{10}$$

where

$$\begin{aligned} f(X) &= \exp(x'\alpha) \\ \varepsilon^* &= \ln\left[1 + \frac{h(X)}{f(X)}\varepsilon\right] \\ x &= \begin{Bmatrix} \ln X & \text{for the Cobb-Douglas} \\ (\ln X \ln^2 X) & \text{for the translog} \end{Bmatrix} \end{aligned}$$

where

$$\ln^2 X = [(\ln X_1)^2 \ldots (\ln X_i X_j) \ldots (\ln X_N)^2]$$

As Kmenta [13] has pointed out, linear estimation of (10) in this case generally leads to bias and inconsistency even for α since $E(\varepsilon^*)$ depends on X (unless $h \equiv f$).

Nevertheless, two general approaches to estimation of (7) are possible. The first involves a three-stage estimation procedure suggested by Just and Pope [12] which first estimates f, then h, and finally uses the estimate of h to develop an asymptotically efficient estimate of f.The second method involves iterative calculation of maximum likelihood estimates for α and β simultaneously. Assuming that both f and h follow a log-linear form (e.g., Cobb-Douglas or translog generality),

$$f(X) = e^{x'\alpha} \tag{11}$$

$$h(X) = e^{1/2x'\beta} \tag{12}$$

these two approaches can be briefly outlined as follows.

A Multistage Estimator

Let the sample corresponding to (7) be written as

$$q_t = f(X_t) + h^{1/2}(X_t)\varepsilon_t, \qquad \varepsilon_t \sim iid\,(0, 1),\ t = 1, \ldots, T. \tag{13}$$

Applying the results of Malinvaud [15] it can be shown that nonlinear least squares estimation of

$$q_t = f(X_t) + \tilde{\varepsilon}_t, \qquad E(\tilde{\varepsilon}_t) = 0, \tag{14}$$

produces a consistent estimate of α so long as β and X are such that h is uniformly bounded. Hence, since $\tilde{\varepsilon}_t = q_t - f(X_t) = h(X_t)\varepsilon_t$, is consistently estimated by

$$\hat{\tilde{\varepsilon}}_t = q_t - e^{x'_t\hat{\alpha}} \tag{14*}$$

it seems reasonable to consider a second stage regression equation where $\hat{\tilde{\varepsilon}}_t$ is regressed on $h(X_t)$ along the lines proposed by Hildreth and Houck [11]. With log-linearity of h this can be accomplished using a linear regression equation

$$\log|\tilde{\varepsilon}_t| = 1/2x'_t\beta + w_t. \tag{14'}$$

As in the Hildreth and Houck case, this regression attains consistency for β. Finally, one can form a feasible Aitkens estimator for α in (14). Since consistency for β is sufficient to allow a consistent estimate of the covariance matrix for $\tilde{\varepsilon}$, this third step leads to asymptotic efficiency for α as in the general feasible Aitken's procedures proposed by Zellner [21] and by Gallant [9].

It has recently been proven by Harvey [10], however, that the multistage approach gives inefficient estimates of β in a log-linear case. Assuming normality, he shows that maximum likelihood (*ML*) estimators are twice as efficient as the Hildreth-Houck type of approach (which estimates β in the second stage). Thus, when normality holds, *ML* estimation may be more desirable.

Maximum Likelihood Estimation

In the case that normality applies in (13) with functional definitions given by (11) and (12), the likelihood function can be written as

$$L = (2\Pi)^{-T/2}(\prod_{t=1}^{T} e^{x'_t\beta})^{-1/2} e^{-1/2 \sum_{t=1}^{t} (q_t - e^{x'_t\alpha})^2 e^{-x'_t\beta}}.$$

Thus, maximizing L is equivalent to minimizing the concentrated likelihood function,

$$L^* = \sum_{t=1}^{T} x'_t\beta + \sum_{t=1}^{T} (q_t - e^{x'_t\alpha})^2 e^{-x'_t\beta}. \tag{15}$$

The concentrated likelihood function is not readily optimized by modification of computerized nonlinear least squares procedures because of the présence of the Jacobian [the first right hand term in (15)] although the same general gradient and search principles are applicable. A general procedure which can be used in problems such as this, however, is the iterative method of scoring (see Harvey for an application of scoring to the multiplicative heteroscedastic case). That is, suppose

$$\gamma = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

and $\hat{\gamma}_j$ is used to denote the estimate of γ at iteration j. Then the estimate $\hat{\gamma}_j$ produced by the iteration

$$\hat{\gamma}_{j+1} = \hat{\gamma}_j + \left[-E\frac{\partial^2 L^*}{\partial\gamma\partial\gamma'}\right]^{-1}_{\gamma = \hat{\gamma}_j}$$

converge to the appropriate ML estimator so long as the initial estimate of γ, say $\hat{\gamma}_0$, is consistent (which would be the case if the multistage estimator above is used as a starting point). In this case, however, the information matrix is block diagonal and, in point of fact,

$$E\left[\frac{\partial^2 L^*}{\partial\gamma\partial\gamma'}\right] = \begin{bmatrix} \frac{\partial^2 L^*}{\partial\alpha\partial\alpha'} & \frac{\partial^2 L^*}{\partial\alpha\partial\beta'} \\ \frac{\partial^2 L^*}{\partial\beta\partial\alpha'} & \frac{\partial^2 L^*}{\partial\beta\partial\beta'} \end{bmatrix} = \begin{bmatrix} \sum_{t=1}^{T} e^{x'_t(2\alpha-\beta)} x_t x'_t & 0 \\ 0 & \frac{1}{2}\sum_{t=1}^{T} x_t x'_t \end{bmatrix}.$$

Hence, consistent and asymptotically efficient estimation of both α and β can be accomplished by alternating iterations

$$\hat{\alpha}_{j+1} = \hat{\alpha}_j - \left[\sum_{t=1}^{T} e^{x'_t(2\hat{\alpha}_j-\hat{\beta}_j)} x_t x'_t\right]^{-1} \left[\sum_{t=1}^{T} e^{x'_t(\hat{\alpha}_j-\hat{\beta}_j)}. (q_t - e^{x'_t\hat{\alpha}_j}) x_t\right] \tag{15'}$$

$$\hat{\beta}_{j+1} = \hat{\beta}_j + \left[\sum_{t=1}^{T} x_t x'_t\right]^{-1} \left\{\sum_{t=1}^{T}\left[1 - (q_t - e^{x'_t\hat{\alpha}_j})^2 e^{-x'_t\hat{\beta}_j}\right] x_t\right\} \tag{16}$$

with consistent starting values possibly given by the multistage procedure.

It has further been observed that the iterative scoring procedure attains asymptotic efficiency after only one iteration (with both (15′) and (16)) when consistent starting values are used for the parameters involved [10]. This suggests several ways in which the Multistage and ML procedures above can be combined into an asymptotically efficient four stage procedure. The two possibilities are given as follows:

Stage 1. Nonlinear least squares estimation of (14) producing a consistent estimate, say $\hat{\alpha}$.

Stage 2. Linear least squares regression of log $|\hat{\varepsilon}|$ on 1/2 ln X_t according to equation (14′) where $\hat{\varepsilon}_t$ is given by (14*). This produces a consistent estimate, say $\hat{\beta}$.

Stage 3. At this stage asymptotic efficiency can be attained for α either by computing a feasible Aitken's estimator for equation (14) as in the multistage procedure or by setting $\hat{\alpha}_j = \hat{\alpha}$ and $\hat{\beta}_j = \hat{\beta}$ in equation (15).

Stage 4. Finally, asymptotic efficiency for β can be attained by substituting $\hat{\alpha}$ for $\hat{\alpha}_j$ and $\hat{\beta}$ for $\hat{\beta}_j$ in equation (16).

Alternative Procedures

For controlled experiments, it is often possible to estimate variance of output at various levels of the variable input. These variance estimates have then been regressed against polynomials in the input to determine its effect on output variability (Kmenta [13, p. 256], Anderson [1]). From the foregoing analysis, however, such an approach apparently attains efficiency for neither α nor β even asymptotically. This is because the information in the sample for one parameter vector is not taken into account in estimating the other. As described above, the multistage procedure attains asymptotic efficiency for α only in the third stage where sample information on β is considered. In the *ML* approach, asymptotic efficiency is attained for α (β) only by considering sample information in relation to the other parameter vector β (α).

Summary

In conclusion, the effect of risk aversion on factor use is not positive for all inputs in general but rather depends on the nature of the production process and the input involved. However, functional forms often used (such as (4)) for investigation of behavior lack this flexibility and always imply the result that a risk averter uses less of each factor than the risk-neutral firm. It thus seems that the greater generality provided in (7) is desirable for both theoretical and empirical work. On the basis of initial examinations, estimational procedures are readily available for estimation of (7), although they are somewhat more complicated than traditional methods.

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