Indivisibility and Divisibility in Land Development Decisions Over Time and Under Uncertainty

by

Amitrajeet A. Batabyal

Department of Economics, RIT, 92 Lomb Memorial Drive, Rochester, NY 14623-5604. Internet aabgsh@rit.edu

and

Seung Jick Yoo

Korea Energy Economics Institute, 665-1 Naeson2-dong, Euiwang-si, Kyonggi-do, 473-713, Republic of Korea. Internet: sjyoo@keei.re.kr

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Abstract

The quasi-option value (QOV) literature originated by Arrow and Fisher (1974) and by Henry (1974) is largely concerned with the analysis of two-period models of land development. Our paper extends this literature by analyzing two scenarios in which the decision to develop land is made in a *multi-period* and *stochastic* framework. In the first scenario, the development decision is *indivisible*. In contrast, in the second scenario, the development decision is *divisible*. Specifically, we study the properties of the indivisible development decision when there is a time constraint on when land is to be developed. We then analyze the ways in which the divisible land development decision depends on the extent of a landowner's landholding and on the number of development opportunities awaiting this landowner.

Keywords: Divisible, Dynamic, Indivisible, Land Development, Time Constraint, Uncertainty

JEL Codes: Q24, R19, D81

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1. Introduction

In two seminal papers, Arrow and Fisher (1974) and Henry (1974) introduced the concept of quasi-option value (QOV) to natural resource and environmental economists. The so called Arrow-Fisher-Henry (AFH) concept of quasi-option value tells us that when land development is both indivisible and irreversible, a landowner who disregards the possibility of procuring new information about the effects of such development will invariably underestimate the benefits of preservation and hence skew the binary choice develop/preserve decision in favor of development.

This elegant and powerful result has been shown to hold in its most general form in a two-period model. Therefore, it makes sense to study two specific issues about the nature of the land development decision in a n-period and stochastic model in which $n>2$. First, it would be useful to analyze the properties of the land development decision when there is a time constraint on when land is to be developed and when this decision is *indivisible*. Indivisible means that the landowner must develop all of his land or none of it and partial land development is not possible. Second, it would be instructive to study the characteristics of the land development decision when this decision is *divisible*. In this paper, divisible means that it is possible for our landowner to partially develop his land.

The first issue of the foregoing paragraph has been addressed by previous researchers. Markusen and Scheffman (1978), Arnott and Lewis (1979), and Capozza and Helsley (1989) have all studied this issue in a deterministic environment. However, when the relevant development decision is irreversible, the use of a certainty framework is inappropriate. In fact, the investment under uncertainty literature (see Pindyck (1991), Dixit and Pindyck (1994), and Hubbard (1994)) tells us that if we are to comprehend the nature of land development decisions in the presence of irreversibilities, then it is essential that we clearly account for uncertainty.

Recently, Titman (1985), Capozza and Helsley (1990), and Batabyal (1996, 1997, 2000) have examined the question of land development over time and under uncertainty. The bulk of Titman's (1985) analysis is carried out in a two-period model. Therefore, this paper does not really address the multi-period nature of the land development problem. Capozza and Helsley (1990) use the concept of a "first hitting time"[1] to show that land ought to be converted from rural to urban use at the first instance in which the land rent exceeds the reservation rent. In contrast with the Cappoza and Helsley (1990) approach, Batabyal (1996, 1997) has used the theory of Markov decision processes to provide discrete-time and continuous-time analyses of the *n*-period land development problem. In both papers, a particular stopping rule is used to ascertain when a stochastic "revenue from development" process ought to be halted. In Batabyal (2000), the "When do I develop land" question is answered by studying the decision problem of a landowner who wishes to maximize the probability of accepting the best possible offer of development, given that these offers are received sequentially over time.

Unlike the indivisible land development question over time and under uncertainty, the divisible land development question has received less attention from previous researchers. Epstein (1980), Hanemann (1989), and more recently Batabyal (1999), have studied alternate aspects of this question. These researchers have all shown that when the land development decision is divisible, the AFH result will not hold in general.

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[1] For additional details on this notion, see Dixit and Pindyck (1994, pp. 83-84) and Ross (1996, pp. 363-366).

This review of the extant literature tells us that although previous studies have analyzed aspects of indivisible and divisible land development decisions, there are interesting and salient questions that have not yet been explored.[2] For instance, in a probabilistic sense, is it desirable to impose a time constraint on the land development decision? Second, how does the presence of a time constraint about when land is to be developed affect the *indivisible* land development decision? Third, what are the statistical properties of a specific random variable that is intimately connected with the indivisible land development decision? Fourth, how do the size of a landowner's landholding and the number of development opportunities that await him influence the *divisible* land development question? The purpose of this paper is to use a dynamic and stochastic framework to shed light on these four hitherto unanswered questions.

The rest of this paper is organized as follows. Section 2 first delineates an intertemporal and stochastic framework and then it uses this framework to answer the first three questions posed in the previous paragraph. Section 3 adapts a model originally due to Derman *et al.* (1975) and answers the fourth question of the foregoing paragraph. Finally, section 4 concludes and offers suggestions for future research.

2. Indivisible Land Development

2.1. Preliminaries

Consider a landowner who owns a plot of land. In this section, consistent with the AFH tradition, we suppose that the development decision is indivisible. As explained in section 1, this means that partial development of the plot of land is not possible. The landowner operates in a

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2 Also see the second paragraph of this section on p. 3.

dynamic and stochastic setting. The setting is stochastic because the decision to develop depends essentially on the receipt of offers to develop land. These offers $O_1, O_2, O_3, \ldots$ are dollar-valued and the offers themselves constitute a sequence of independent, identically distributed (iid) continuous random variables. The setting is dynamic because the offers are received over time. Our analysis begins at time $t=1$. This means that O_1 is the offer received when our analysis begins, O_2 is the second offer, O_3 is the third offer, and so on and so forth.

As indicated in section 1, a key feature of our analysis is the fact that we study the land development decision when there is a *time constraint* on when land is to be developed. We model this aspect of the problem by supposing that our landowner will develop his land at a predetermined time in the future, say, time t. To comprehend this time constraint, consider the following situation. A landowner who is a farmer may decide that agriculture is not a sufficiently desirable career prospect and that he would hence like to develop his agricultural land and, in the process, convert it to urban land. However, because there are costs associated with such a transition, our landowner cannot develop his land immediately. Therefore, he decides that he would like to develop his land at some future time, say, time t.

Given this state of affairs, we now need to ponder an important issue that has two aspects to it. Looking at the first aspect, is it a good idea to impose such a time constraint on the land development decision? To answer this question, we shall compute the likelihood that the largest offer (in dollar terms) to develop land will be received by our landowner at the predetermined time t. Looking at the second aspect, the reader should note that the presence of this kind of time constraint fundamentally alters the nature of the land development decision. In particular, unlike the cases studied by Cappoza and Helsley (1990), by Batabyal (1996, 1997, 2000), and by others, our landowner's decision problem now is *not* to determine when to develop his plot of land, given that he is faced with the stochastic process of dollar-valued offers $O_1,O_2,O_3,...$ Instead, our landowner's role now is considerably more passive. Specifically, given the predetermined development time, all that our landowner can do—and this connects with the first aspect that we have just discussed—is to compute the likelihood that the offer at time t, O_t, is larger than each of the previous offers $O_1,...,O_{t-1}$. To facilitate the computation of this likelihood, we shall say that O_t is a *record* if $O_t>\max(O_1,...,O_{t-1})$. Mathematically then, our landowner is interested in calculating the probability that a record occurs at time t. As such, we now first compute $Prob\{record\ occurs\ at\ time\ t\}$ and then we shed light on certain statistical properties of the *record* random variable.

2.2. Analysis

We know that the dollar-valued offers $O_1,O_2,O_3,...$ are independent, identically distributed, and continuous random variables. Therefore, it follows that with probability one, the individual offers all have different values. Hence, the largest of $O_1,O_2,...,O_t$ is equally likely to be either O_1, or O_2,..., or O_t. Now, because there is a record at time t when O_t is the largest offer, we conclude that

$$Prob\{record\ occurs\ at\ time\ t\}=\frac{1}{t}. \tag{1}$$

We can now address the question of the desirability of a time constraint on the land development decision. To this end, suppose that, *ceteris paribus*, our landowner would like to develop his land upon receipt of the largest offer of development. In such a situation, equation (1) tells us that, probabilistically speaking, the likelihood that our landowner will get the largest offer of development is given by the reciprocal of the predetermined development time t. Hence, the further away is this predetermined development time, the less likely it is that our landowner will receive the largest offer. Clearly, equation (1) tells us that it is *not* desirable for our landowner to wait a long time before developing his land. However, this finding does not necessarily mean that t should be set very small. To see this, note that setting t=2 implies that our landowner will develop his land with the highest offer with probability 0.5. However, because there is no necessary relationship between the timing of an offer and its dollar value, there is presumably some value to not developing land very early in the planning horizon.

It should be clear to the reader that the number of records by time t is a random variable. Consequently, we can ask what its expectation and variance are. To compute the expectation, $E[number\ of\ records\ by\ time\ t]$, let us first define the indicator variable I_j such that

$$I_j = \begin{cases} 1 \ \ if\ a\ record\ occurs\ at\ j \\ 0\ \ otherwise \end{cases} \tag{2}$$

With this definition in place, it is easy to see that $E[\sum_{j=1}^{j=t} I_j] = \sum_{j=1}^{j=t} E[I_j] = \sum_{j=1}^{j=t} 1/j$. From this, we can deduce that

$$E[number\ of\ records\ by\ time\ t] = \sum_{j=1}^{j=t} \frac{1}{j}. \tag{3}$$

To compute $Var[number\ of\ records\ by\ time\ t]$, it is helpful to recognize that the $I_1, I_2, ..., I_t$ are independent random variables. Specifically, for $j<k$, we have $Prob\{I_j=1/I_k=1\} = Prob\{I_j=1\}$. This result holds because knowing that O_k is the largest of $O_1,...,O_j,...,O_k$ tells us nothing about whether or not O_j is the largest of $O_1,...,O_j$. Now, using this line of reasoning, we can tell that

$Var[\sum_{j=1}^{j=t} I_j] = \sum_{j=1}^{j=t} Var[I_j] = \sum_{j=1}^{j=t} [(1/j)\{(j-1)/j\}]$. Therefore, we have

$$Var[number\ \ of\ \ records\ \ by\ \ time\ \ t] = \sum_{j=1}^{j=t} \frac{j-1}{j^2}. \tag{4}$$

Equation (3) tells us that the expected number of records is a function of the time at which our landowner would like to develop his land. In particular, as this time increases, the expected number of records also increases but at a decreasing rate. Equation (4) gives us a standard measure of dispersion for the random variable denoting the number of records by time *t*. Specifically, equation (4) tells us that the variance of the number of records by time *t* is a complicated (non-linear) function of the time at which our landowner would like to develop his land.

Finally, note that because the first record can occur at any arbitrary time, it is of considerable interest to analyze some properties of the random variable T, where $T=\min\{t\text{: } t>1,\ record\ occurs\ at\ time\ t\}$. In words, T is an index for the first instance at which a record occurs. To obtain additional insight into the nature of the land development decision in the presence of a time constraint, let us now compute the expectation of T, $E[T]$, and the probability that T is finite, i.e., $Prob\{T<\infty\}$. To compute $E[T]$, it is helpful to recognize that the event $T>t$ occurs if and only if O_1 is the largest of the offers $O_1, O_2, \ldots, O_t$. Using this piece of information, we can infer that

$$E[T] = \sum_{t=1}^{t=\infty} Prob\{T>t\} = \sum_{t=1}^{t=\infty} \frac{1}{t} = \infty. \tag{5}$$

Now note that $Prob\{T=\infty\} = \lim_{t\to\infty} Prob\{T>t\}$. Therefore, using equation (5), we can evaluate this limit. We get $\lim_{t\to\infty} Prob\{T>t\} = \lim_{t\to\infty} 1/t = 0.$ This allows us to conclude that

$$Prob\{T<\infty\} = 1. \quad (6)$$

Equations (5) and (6) together provide us with interesting information about land development in a dynamic and stochastic framework in which there is a time constraint on development. We see that even though it is certain that the first record will occur at some finite time (equation (6)), on average, the first instance at which a record occurs is infinity (equation (5)). Put differently, equation (5) tells us that as long as the time at which our landowner would like to develop his land is finite, he cannot go wrong with this time constraint in the sense that, on average, the first instance at which a record will occur is infinity. Recently, using a very different theoretical model, Batabyal and Yoo (2003) have shown that even though the probability of land development is always positive, the expected wait until the land in question is developed is infinity. Consequently, in an expected waiting time sense, a landowner will never develop (always preserve) his land. The reader will note that the findings contained collectively in equations (5) and (6) are similar in nature to this Batabyal and Yoo (2003) result. We now study the divisible land development decision over time and under uncertainty.

3. Divisible Land Development

3.1. Preliminaries

To study the divisible land development decision effectively, in this section, we suppose that our landowner has $\hat{L}$ acres of land available for possible development. Now, in each of T time periods, an offer to develop land will be received by our landowner with probability p. As in section 2, we assume that these offers are independent random variables. When an offer to develop land is made, our landowner must decide how many acres of his remaining land to develop. If he agrees to develop d acres, then he receives benefits given by the function $B(d)$. In this section, there is no time constraint on land development. Consequently, our landowner's decision problem is to determine how many acres of land to develop when presented with offers over time. His objective is to maximize the expected sum of benefits from land development.

3.2. Analysis

In the remainder of this section, we suppose that the benefit function $B(d)$ is a non-decreasing and concave function of d with $B(0)=0$. Let $R_t(L)$ denote the maximal expected additional return achievable when there are t time periods remaining, L acres of land available for development, and an offer to develop land has just been received by our landowner. Then, as shown in Derman *et al.* (1975) and Batabyal (1999), the return function $R(\cdot)$ satisfies an optimality equation. That equation is

$$R_t(L)=\max_{d\in[0,L]}[B(d)+\hat{R}_{t-1}(L-d)], \tag{7}$$

where $t>0$, $R_0(L)=0$, and

$$\hat{R}_s(L)=\sum_{i=0}^{i=s} p(1-p)^i R_{s-i}(L). \tag{8}$$

In words, $\hat{R}_s(L)$ is the maximal expected sum of returns given that L acres of land remain for possible development, there are s time periods remaining, and the landowner does not know whether an offer is forthcoming.

Now, if we condition on the receipt (or non-receipt) of an offer, then we get

$$\hat{R}_s(L)=pR_s(L)+(1-p)\hat{R}_{s-1}(L). \tag{9}$$

We know that the benefit function $B(\cdot)$ is concave by assumption. What does the concavity of $B(\cdot)$ mean for the concavity of the return function $R(\cdot)$? The answer is contained in

PROPOSITION 1: $R_t(L)$ is concave in L.

PROOF: The proof proceeds by induction on t. We know that $R_1(L)=B(L)$ is concave. As such, assume that $R_i(L)$ is concave in L for $i=1,...,t-1$. To show that $R_t(\cdot)$ is concave, we have to show, for $\theta\in(0,1)$, that

$$R_t\{\theta L_1+(1-\theta)L_2\}\geq\theta R_t\{L_1\}+(1-\theta)R_t\{L_2\}. \tag{10}$$

Now, for $d_1\leq L_1$ and $d_2\leq L_2$, we have $R_t\{L_1\}=B(d_1)+\hat{R}_{t-1}\{L_1-d_1\}$ and $R_t\{L_2\}=B(d_2)+\hat{R}_{t-1}\{L_2-d_2\}$. Further, because $\theta d_1+(1-\theta)d_2\leq\theta L_1+(1-\theta)L_2$, from equation (7) it follows that $R_t\{\theta L_1+(1-\theta)L_2\}\geq$ $B\{\theta d_1+(1-\theta)d_2\}+$ $\hat{R}_{t-1}\{\theta(L_1-d_1)+(1-\theta)(L_2-d_2)\}\geq$ $\theta B(d_1)+(1-\theta)B(d_2)+$ $\theta\hat{R}_{t-1}\{L_1-d_1\}+$ $(1-\theta)\hat{R}_{t-1}(L_2-d_2)=$ $\theta R_t\{L_1\}+(1-\theta)R_t\{L_2\}$. From this last step, we see that equation (10) is satisfied and hence $R_t(L)$ is concave in L. ■

To enable the reader to fully comprehend the nature of the optimal divisible land development decision, we will now make a definition. To this end, let $d_t(L)$ be the value of d that maximizes the optimality equation (equation (7)). In words, $d_t(L)$ is the optimal amount of land to develop when the land available for possible development is L, there are t time periods remaining, and an offer to develop land has just been received by our landowner. We are now in a position to state the principal result of this section. We have

THEOREM 1: The optimal divisible land development decision rule $d_t(L)$ is non-decreasing in L and non-increasing in t.

PROOF: See theorem 3 in Derman *et al.* (1975, pp. 1123-1124). ■

From a practical perspective, theorem 1 contains two salient results for our landowner. First, this theorem tells us that the more land one has available for potential development, the more one should develop. Second, this theorem tells us that the more time one has left for land development—or, put differently, the greater the number of offers one expects to receive—the less land one should develop. These results conform well with our intuition about what an optimal divisible land development decision rule ought to look like. Even so, the reader should note that theorem 1 holds only when the benefit from land development is a concave function of the amount of land developed, i.e., when $B(d)$ is a concave function.[3]

4. Conclusions

In this paper we studied two aspects of land development in an intertemporal and stochastic framework. In section 2, we analyzed land development when the development decision is *indivisible* and when there is a time constraint specifying when land is to be developed. Our analysis showed that the further away the time constraint specifying when land is to be developed, the less likely it is that our landowner will receive the largest offer to develop land. Therefore, it is *not* desirable for our landowner to wait a long time before developing land. In addition, our analysis also showed that as long as the time at which our landowner would like to develop his land is finite, he cannot go wrong with a time constraint in the sense that, on average, the first instance at which a record occurs is infinity.

In section 3, we analyzed two properties of the optimal *divisible* land development decision rule. First, we showed that the more land one has available for possible development, the more one ought to develop. Second, our analysis revealed that the more time one has left for land development—or, in other words, the greater the number of offers one expects to receive—the less land one ought to develop.

The analysis of this paper can be extended in a number of directions. In what follows, we suggest two possible extensions. First, the reader will note that in section 2, we studied a situation

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[3] It is difficult to imagine a benefit function that is convex in d. As such, it is our contention that the case in which $B(d)$ is concave in d is the more interesting and realistic case. In any event, it is possible to use a result in Derman *et al.* (1975) to show that when $B(d)$ is convex in d, it is optimal to develop all land when an offer is received by our landowner.

in which a landowner understands that the stochastic process of offers that he faces is iid in nature. Therefore, it would be useful to analyze a scenario in which the landowner recognizes that the random offers that he faces are independent but not identically distributed over time. Second, in section 3, in each of T time periods, an offer to develop land is received by our landowner with a fixed probability p. Consequently, it would be interesting to analyze a situation in which this probability is not fixed but chosen at random from some cumulative distribution function. Studies that analyze these aspects of the problem will provide additional insights into the properties of indivisible and divisible land development decisions over time and under uncertainty.

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