LINEAR PROGRAMMING—AN EXAMPLE

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This note has two purposes: firstly, to introduce the technique of linear programming to interested readers; secondly, to show that its application is feasible in Australian farm management work. We feel that too many people on coming across econometric techniques in journals say either "it's too complicated" or "the data aren't available in Australia." By carrying through an example for an actual farm it is hoped to dispel some of these fears in relation to linear programming. An extension of the technique is also illustrated.

Budgeting is a well-known and often-used technique in farm management. A series of budgets are set up, each covering an alternative farm programme for some future period—usually the next financial year. The most profitable plan of those studied is then selected. The essential feature of budgeting is thus the comparison of the profits arising from different plans, each being studied separately. To be sure of selecting the most profitable one, every possible plan would have to be budgeted. Obviously this is not feasible in practice. No decision, therefore, can be made about the countless number of unspecified programmes which are not budgeted.

Linear programming (or activity analysis as it is sometimes called) is the formalised or mathematical counterpart of budgeting. It is a continuous, rather than a trial and error method by which is selected the *one* unique production programme, out of *all* possible resource combinations, which yields the maximum profit.[1] Thus, linear programming performs the same function as budgeting. Its advantages over budgeting are that it is more precise, quicker and less cumbersome. In a practical sense it suffers from the disadvantage that, unlike budgeting, its mechanics are not easily explained to farmers. To overcome this, the recommended plan could be given to the farmer in budget form. Both methods use the same data (input-output coefficients and prices) and assumptions (linearity and constant input-output ratios). Where budgeting is possible, so is programming.

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[1] In farm management the programmes studied may relate to the various possible combinations of farm enterprises, e.g., sheep and wheat, or to alternative methods within the one enterprise, e.g., feeding programmes. The example given is of the former type. Programming can also be used for broader problems involving the allocation of scarce resources such as, for instance, the determination of the best allocation of irrigation water among competing areas.

It should, perhaps, be pointed out that linear programming is a special type of production function analysis. Given a fixed quantity of resources programming indicates the best way of using them. Ordinary production function analysis, on the other hand, indicates the best way of combining resources when the quantities available are not fixed.

There are various ways of carrying out the technique of activity analysis. The method presented here[2] is the simplest so far developed but has the disadvantage that it cannot be used for complete analysis where the most profitable organisation of the farm involves more than four or five enterprises. In such cases the original method,[3] involving rather more tedious computations, is best used. However, we think that in Australia the short cut method will be of most use in the immediate situation. The majority of our farms are not suitable for more than four types of production when both managerial capacity and physical suitability are taken into account.

In our example the short-cut method has been used to determine the most efficient combination of enterprises for a farm on the New South Wales south coast. The property, consisting of 923 acres of undulating granite country typical of the Moruya area, is currently being used for dairying and weaner beef raising but would be suitable also for dry sheep. On taking over the property in 1955 the present operators sought assistance from the Department of Agriculture in planning to maximise income from their farm resources.

The data required to solve the problem are listed in Table I which shows the quantity of each resource that is available, the input-output coefficients showing the amount of each resource required to produce one unit of each product, and the net income per unit of product.

The land is of fairly uniform quality. If this were not the case the varying amounts of different quality could be listed as separate resources, as could also be done if there were seasonal variations in carrying capacity. The amount of operating capital likely to be available for the coming year has been estimated from farm records being kept by the operators.[4] Labour has been divided into two categories because its supply is not uniform throughout the year, the operators taking their annual holidays in the winter. Spring-summer-autumn labour availability is based on each partner being prepared to work forty-eight hours per week plus an additional twenty hours per week of paid family labour.

Input-output coefficients have been derived from farm record data in consultation with the operators and Departmental officers. Thus it is estimated that production of 1 lb. of butterfat would require .033 acres of land, 1 lb. of weaner beef .018 acres, and 1 lb. of wool .067 acres.

Net income per lb. of product is obtained by subtracting the relevant operating capital coefficient from the expected gross return per unit of product.[5] The expected gross return per unit of product, because of

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[2] See: Frederick V. Waugh and Glenn L. Burrows, "A Short Cut to Linear Programming", *Econometrica,* Vol. 23, No 1 (January, 1955), p. 18.

[3] See: Earl O. Heady, "Simplified Presentation and Logical Aspects of Linear Programming Technique", *Journal of Farm Economics,* Vol. 36, No. 5 (December, 1954), p. 1035; and James N. Boles, "Linear Programming and Farm Management Analysis", same journal, Vol. 37, No. 1 (February, 1955), p. 1.

[4] Where income is obtained continuously throughout the year (as on dairy farms), this figure poses some problems in contrast to the situation of, say, a grazier who on receiving his annual wool cheque can decide how much production capital he will use in the coming season.

[5] The expected gross returns used were £.225, £.049 and £.275 per 1 lb. of butterfat, weaner beef and wool respectively.

TABLE I

Data for Programming

Resource.	Unit.	Supply of Resource.	Resource Number $i = 1, \dots 4$	Amount of Resource required to produce one lb. of product j. $j = 1, 2, 3$.		
				Butter-fat.	Weaner Beef.	Wool.
				(1)	(2)	(3)
Land...	acres	923	(1)	·033	·018	·067
Operating Capital ...	pounds	3,000	(2)	·127	·023	·150
Spring—Summer—Autumn Labour...	hours	4,524	(3)	·161	·037	·135
Winter Labour ...	hours	1,160	(4)	·041	·009	·035
Net Income per lb. product	pounds	...	(r_j)	·098	·026	·125

price uncertainty, can be only an informed guess (as is also the case in budgeting). Notice that in calculating net return, as listed, no allowance has been made for the operators' own labour, though hired labour cost is included in the operating capital coefficients.

Considering the coefficients for butterfat production it can be seen that if land was the only resource necessary, a maximum of 923/.033 or 27,970 lb. could be produced. However, resources other than land are required. How much operating capital would be required if 27,970 lb. of butterfat were produced? It is known that each lb. requires £.127, hence £3,552 of operating capital would be necessary. Since only £3,000 is available it is obvious that production of butterfat from the land resources is limited by operating capital. Further calculations would show whether labour resources would also limit production from the land. It is this principle which is used implicitly in budgeting.

Waugh and Burrows Short Cut Method of Programming

From the data of Table I a table is derived showing the *proportions* of the available quantities of the various resources which would be needed in order to obtain an arbitrary income, say £*d*, from each enterprise. Considering, for example, operating capital in relation to weaner beef production we wish to find the proportion *of* (A) the amount of such capital required to produce, say, £4,000 net return to (B) the amount of operating capital available, i.e., £3,000. Now, from Table I it is known that £.023 must be spent to obtain £.026 net return. Therefore, to obtain £4,000 net return £(4,000)(.023)/.026 of operating capital would be required. The proportion, (A) to (B), is thus (4,000) (.023)/(.026) (3,000). This figure, 1.1795, is then entered in the operating capital—weaner beef position of the new table being drawn up.

In mathematical language, the a_{ij} matrix in Table I is transformed into a new matrix, b_{ij}, by means of the transformation :—

$$b_{ij} = \frac{a_{ij}d}{s_i r_j},$$

where : a_{ij} is the input-output coefficient given in row i, column j of Table I. (For example, a_{43} is ·035) ;

d is the arbitrary net income chosen ;

s_i is the available supply of resource i ;

r_j is the net return per unit of product j.

The new matrix is presented in Table II. No matter what arbitrary income had been chosen, these b_{ij} values would still follow the same size pattern (both within and between columns). Consider the figures in the butterfat column. They tell us that to obtain £4,000 net return, 1·4593 times the available amount of land, 1·7279 times the available operating capital, 1·4526 times the available spring–summer–autumn labour and 1·4426 times the available winter labour would be required. This indicates that for butterfat production, operating capital is the limiting resource.

Likewise, the largest figure in each of the other enterprise columns indicates the limiting factor for that enterprise. For both weaner beet and wool it is land. In Table II the maximum values have been marked with an asterisk. Comparison of these maximum values enables us to select the most profitable single enterprise. It is that which has the smallest maximum value—in this case butterfat.

The reasoning behind this conclusion is as follows: for each enterprise the limiting resource is indicated by the column maximum. Hence, for each enterprise the maximum income attainable if the farm resources were devoted only to that enterprise is, by simple arithmetic, the arbitrarily selected income, £d, divided by the column maximum. Thus, in butterfat production where operating capital is the limiting input, production of £4,000 net return requires 1.7279 times the amount of capital available. Therefore the net income obtainable from one times the capital available is £4,000/1.7279 or £2,315.

TABLE II

Proportion of Available Resources

Necessary to Produce £4,000 of Net Income

Resource.	Enterprise.		
	Butterfat.	Weaner Beef.	Wool.
Land	1·4593	3·0002*	2·3229*
Operating Capital	1·7279*	1·1795	1·6000
Spring—Summer—Autumn Labour	1·4526	1·2582	·9549
Winter Labour	1·4426	1·1936	·9655

* Maximum value in each column indicating the limiting resource for each enterprise.

In the case of weaner beef and wool the maximum net return possible when the farm is devoted entirely to either would be £4,000/3.0002 and £4,000/2.3229, respectively. Obviously, the most profitable single enterprise is the one with the lowest column maximum. In other words, the minimax, i.e., the smallest of a given set of maximum values, tells us the most profitable single enterprise.

Having decided that butterfat would be the most profitable single enterprise, it is easy to calculate the amount of butter to be produced since the total net return (£2,315) and net return per lb. of butterfat (£.098) are known. It is 2,315/.098 or 23,622 lb. butterfat. On the basis of 185 lb. butterfat per cow, the recommended number of cows would be 23,622/185 or 128.

However, a closer study of Table II indicates that a combination of two or more of the enterprises would be more profitable than butterfat alone. The matrix column minimax is in the capital row. Compare the capital coefficients of weaner beef and wool with the minimax. They are both smaller. That is, weaner beef and wool both use operating capital more efficiently than does butterfat production. It follows that it would be more profitable to combine butterfat production with weaner beef or wool than to have dairying only.

To determine the best combination of two processes, graphs are used. In each graph of Figure I the b_{ij} coefficients from Table II are plotted on the vertical axes, one enterprise on the left and the other on the right. Corresponding resource (i) values are joined by straight lines. These are termed resource requirement lines. The horizontal axis measures the proportion of each enterprise in the combination. Thus in the centre graph, butterfat coefficients (b_{i1} values) are marked on the left vertical axis and wool (b_{i3}) on the right. At the extreme left of the horizontal axis the net return is derived wholly from butterfat. At the mid-point, net return derivation would be 50 per cent from dairying and 50 per cent from wool.

The limiting resource for any enterprise combination is indicated by the highest resource requirement line cut by a vertical line drawn from the relevant enterprise combination position on the horizontal axis. Thus a 50:50 combination of dairy and beef production would be limited by land and from the graph it can be read that such a combination producing £4,000 net return would require 2.24 times the amount of land available. The income resulting from use of the 923 acres would be 4,000/2.24 or £1,786. A 95 per cent butterfat: 5 per cent beef combination would be limited by capital. Maximum net return attainable from this combination would be 4,000/1.7 or £2,353. Obviously the combination yielding the maximum net return is indicated by the lowest point on the composite uppermost line. (The logic behind this is the same as that explained in relation to the column minimax of Table II except that two dimensions are now involved). The lowest points have been marked A, B and C on the graphs, A being the lowest of the three.

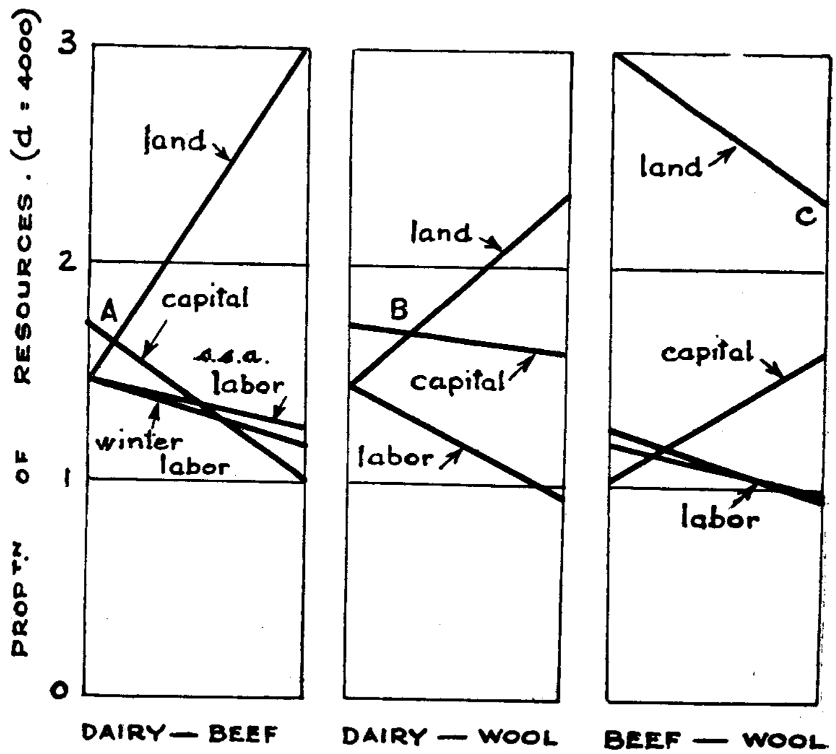

Fig. I.—Graphical determination of best combination of two enterprises.

The combinations and feasible net returns attainable at each of these points are:

	£
A—13 per cent beef: 87 per cent dairy	2,424
B—26 per cent wool: 74 per cent dairy	2,381
C—0 per cent beef: 100 per cent wool	1,724

Assuming 185 lb. of butterfat per dairy cow and 279 lb. of weaner beef per breeding cow after allowing for calf loss, the recommended combination of two enterprises to attain maximum net return would be 116 dairy milkers and 44 beef breeders.

Would some combination of the three enterprises be still more profitable? A yes or no answer can be given to this question by another small piece of graphic analysis. If such analysis shows the answer to be yes, then the best combination can be derived by matrix algebra, simply explained by Waugh and Burrows in the original reference. In this example, however, the addition of the third enterprise, wool, would not increase net returns. This is indicated by Figure 2 in which the enterprises are plotted by measuring the limiting factor for butterfat on one axis and the limiting resource for weaner beef on the other. These two enterprises are chosen because they figure in the most profitable two-enterprise combination. Having plotted the enterprises, there being only three in this example, those figuring in the best combination of two are joined by a straight line. Hence, in Figure 2, butterfat and weaner beef are joined. If the plotted point for any other enterprise lies below this line it is possible to increase net income by adding that enterprise.[6] In this example, wool lies above the butterfat-beef line and therefore cannot help to use available resources more effectively.

The farmers' problem has thus been solved. The one unique combination of enterprises that will yield maximum net return is that involving 44 beef breeders and 116 dairy milkers.[7] With this programme the operators of the South Coast farm will use all of their available land and operating capital but will have some labour unused. The amount

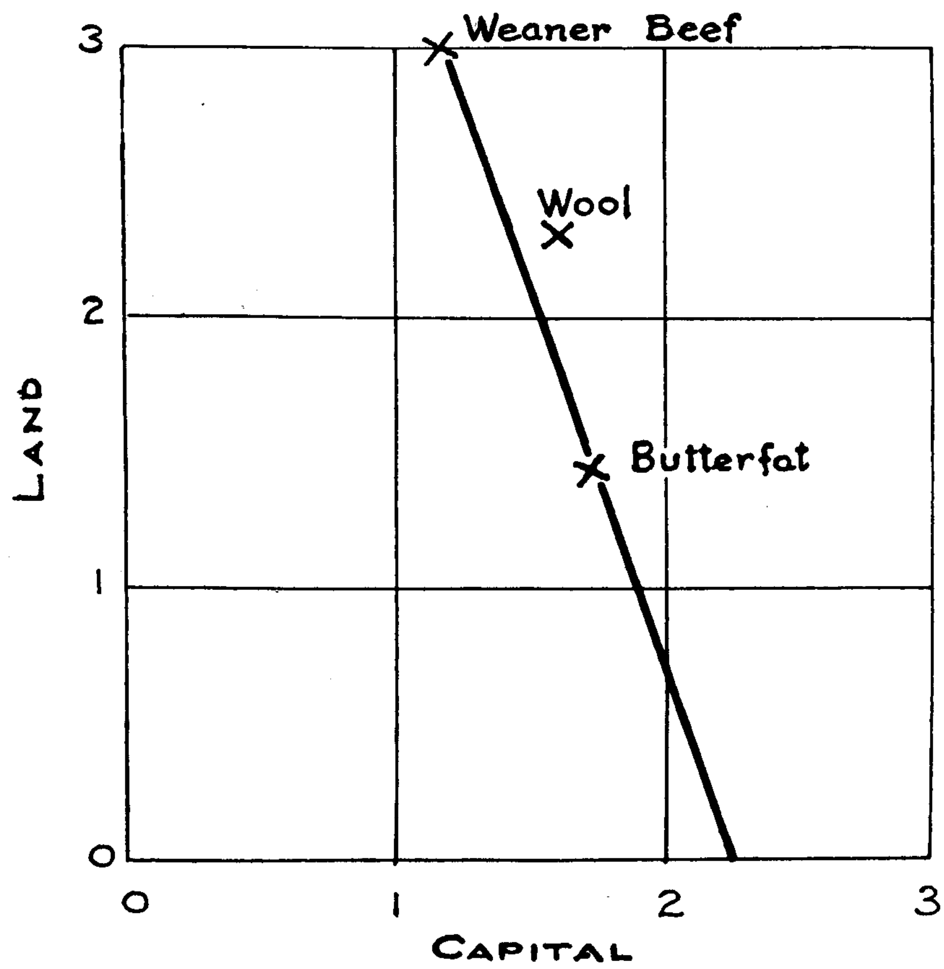

Fig. 2.—Graph showing whether additional enterprises should be added.

[6] The logic behind this statement is again rather similar to that explaining why the minimax of Table II indicates the best single enterprise except that it is rather more complicated because three dimensions are involved.

[7] Obviously, the accuracy of this conclusion depends upon the quality of the data used—as also applies to the budget approach which has the additional disadvantage of not leading quickly or precisely to the best resource combination.

of labour unused can be calculated from a consideration in conjunction with the best enterprise combination, of the labour input coefficients given in Table I. Since the farm offers scope for developmental work, this labour would not be wasted. If there were no developmental opportunities for its use, consideration could be given to using the excess labour by incorporating some labour intensive enterprise (i.e., one where labour would be the limiting factor—such as poultry meat production) into the farm programme.

Price Changes

The most profitable farm programme, under the restrictions imposed by a given set of price and input-output relationships and resource quantities, has been found. Of the restricting factors, product prices, rather than input prices and input-output coefficients, are most likely to change markedly in the short run, i.e., in the period programmed. Naturally, it is of interest to the farmer to have some idea of the extent to which the prices of his alternative products can rise or fall without affecting the most profitable enterprise combination. We have devised a simple method for finding these product price limits, applicable when the short-cut method of programming has been used. As an example, the price to which (*a*) wool must rise or (*b*) weaner beef must fall for wool to displace beef from the most efficient combination of enterprises, assuming other factors remain constant, has been determined. The method illustrated would also apply, with simple modifications, if changes in other factors were being considered.

Change in Most Profitable Single Enterprise

The minimax of Table II indicates dairying to be the most profitable single enterprise. At what price would wool be the most profitable enterprise? In other words, the price at which the minimax, 1.7279, also occurs as the land coefficient for wool, has to be determined. This price, at which both enterprises would be equally profitable, can be found from the transformation

$$b_{ij} = \frac{a_{ij}\, d}{s_i\, r_j},$$

used previously, except that r_j is now the unknown, b_{ij} being 1·7279. Having found r_j, which in this case is (·067) (4,000)/(923) (1·7279) or a net return of £·168 per lb., the equalising gross return for wool is £·318 per lb. since each lb. produced has an associated cost of £·150. Thus, assuming no other changes, the price of wool would have to rise by at least 10·3 pence above its estimated price of 5s. 6d. for it to be the most profitable single enterprise.

Change in Most Profitable Combination of Enterprises

However, the price of wool would not have to rise to this extent to displace weaner beef from the most profitable two-enterprise programme. This is obvious from Figure 1, where A is only .05 units lower than B. If A and B had been the same height above the horizontal axes, the relevant combinations of dairying with beef and dairying with wool would both have been profit maximisers. Hence the wool price at which

B, indicating the most profitable dairy-wool combination, is the same height as A above the horizontal axis has to be determined. This involves two steps. Firstly, the finding of new b_{13} and b_{23} values for which the land and capital resource requirement lines of the dairy-wool graph of Figure 1 intersect at a height of 1.65 units above the horizontal axis. Secondly, having found these values (one is sufficient), the wool price related to them can be determined from the transformation as illustrated above.

The new b_{13}, b_{23} values can be found by the use of an equation derived from analytical geometry. Consider Figure 3, which corresponds to the dairy-wool graph of Figure 1, except that only the resource requirement lines of the limiting resources—land (RS) and capital (QT)—have been drawn. B is their point of intersection, as in Figure 1, at a height of 1.7 units. RS′ and QT′ are the resource requirement lines, applicable at some unknown wool price, such that their intersection point, B′, is the same height above OX, 1.65 units, as A of Figure 1. The length of OX is assumed to be one unit.

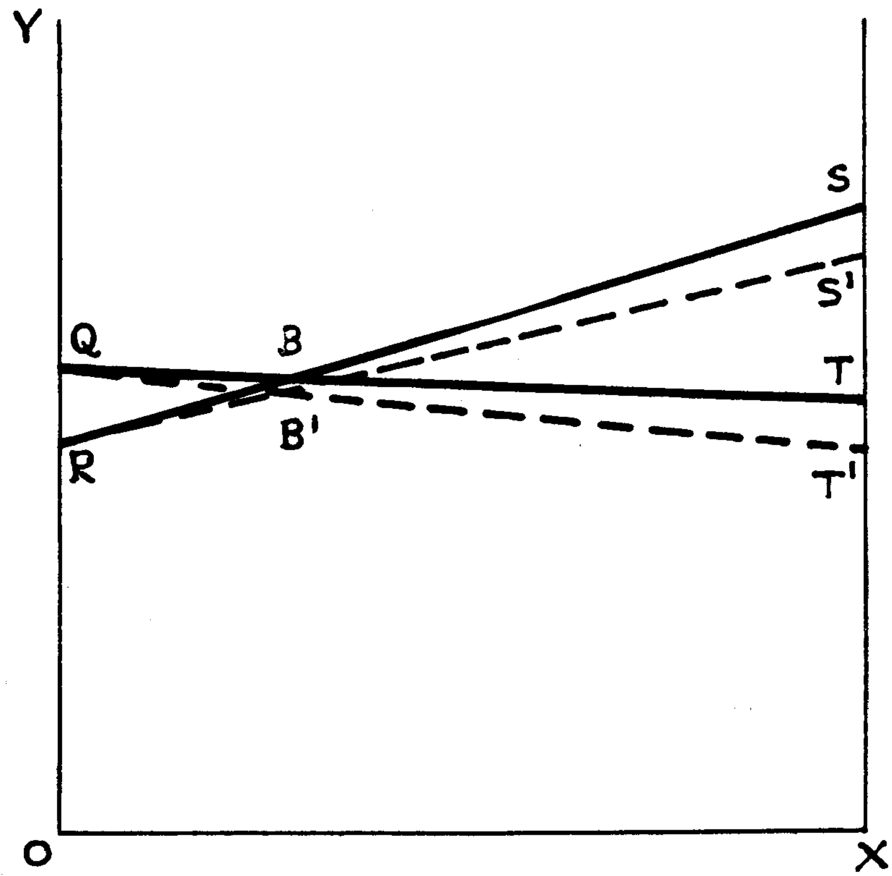

Fig. 3.—Resource requirement lines, RS′ (land) and QT′ (capital), at an unknown wool price such that a combination of dairy and wool enterprises would just be as profitable as the best combination of dairy and beef enterprises.

Co-ordinates : Q : (0, q) R : (0, r)
S : (1, s) T : (1, t)
S′ : (1, s′) T′ : (1, t′).

The first step is to find the position of S′ and T′ so that B′ is the required height above OX. In other words, it is required to find the y coordinates of S′ and T′.

We know—

(i) the y coordinate of B′, *i.e.*, its height above the horizontal axis;

and (ii) that $\frac{s'}{t'} = \frac{s}{t}$.

From the general equation for a line joining two given points we can obtain the equation for QT′ and RS′. They are respectively :—

$$x = \frac{y - q}{t' - q} \text{ and } x = \frac{y - r}{s' - r}.$$

The y coordinate of B′, the point of intersection of these two lines, is found by solving these two equations for y. Hence—

$$y = \frac{qs' - rt'}{s' - r - t' + q}$$

But $$\frac{s'}{t'} = \frac{s}{t}, \text{ so that } s' = \frac{st'}{t}.$$

Thus, on substituting for s',

$$y = \frac{\frac{qst'}{t} - rt'.}{\frac{st'}{t} - r - t' + q}.$$

In this equation t' is the only unknown. It may be rewritten :

$$t' = \frac{yt\ (q - r)}{s(q - y) + t(y - r)}.$$

From the b_{ij} matrix of Table II, q, r, s and t are obtained whilst y is the height of the lowest minimum point on the composite uppermost lines of Figure 1.

In our example :

$$t' = \frac{(1{\cdot}65)\ (1{\cdot}6)\ (1{\cdot}7279 - 1{\cdot}4593)}{(2{\cdot}3229)\ (1{\cdot}7279 - 1{\cdot}65) + (1{\cdot}6)\ (1{\cdot}65 - 1{\cdot}4593)}$$
$$= 1{\cdot}4588.$$

Substituting this value into the transformation equation, the unknown, net return per lb. of wool, can be determined, viz.:

$$1{\cdot}4588 = \frac{(4000)\ ({\cdot}15)}{(3000.)\ (r_3)},$$

hence $r_3 = £{\cdot}137$.

The relevant gross return per lb. of wool is therefore £(.137 + .150) or £.287. That is, other factors remaining constant, if the price of wool rose by more than 3 pence per lb., i.e., £(.287 — .275), it would be more profitable to combine dairying with wool rather than with beef production.[8]

Likewise it can be determined that weaner beef prices would have to fall by more than a halfpenny per lb. or 16 shillings per average weaner for wool to displace beef from the profit maximising enterprise combination.

In cases where the composite uppermost lines of the Figure 1 graphs contain more than one intersection point, the formula must be applied as many times as there are intersection points because, as the product price changes, one of the other intersection points may become the minimum point. The one which reaches the relevant height above the horizontal axis first is that which has the smallest associated price change.

[8] A combination of the three enterprises would not be more profitable because weaner beef and wool both have the same limiting resource, land.