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# MATHEMATICAL APPENDICES FOR: RECONCILING THE VON LIEBIG AND DIFFERENTIABLE CROP PRODUCTION FUNCTIONS

by

Peter Berck and Gloria E. Helfand



DEPARTMENT OF AGRICULTURAL AND RESOURCE ECONOMICS

BERKELEY

CALIFORNIA AGRICULTURAL EXPERIMENT STATION

*University of California*

DEPARTMENT OF AGRICULTURAL AND RESOURCE ECONOMICS
DIVISION OF AGRICULTURE AND NATURAL RESOURCES
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California Agricultural Experiment Station
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## Appendix I

This appendix will show that increased dispersion in the inputs lowers output. Let $F_1(\varepsilon_1)$ and $H_1(\varepsilon_1)$ be CDFs for $\varepsilon_1$ and let $Y_{F_1}$ and $Y_{H_1}$ be the aggregate output when $\varepsilon_1 \sim F_1$, and $\varepsilon_1 \sim H_1$, respectively.

*Theorem*: If $H_1$ is more risky than $F_1$ in the sense of a mean-preserving spread, then $Y_{F_1} \geq Y_{H_1}$.

*Proof*: Let $z' = (z_1', z_2', z_3')$ and $z'' = (z_1'', z_2'', z_3'')$ be two different 3-vectors. Yield is $y(z) = \min(z_1, z_2, z_3)$. For $t \in [0, 1]$,

$$y[z't + z''(1 - t)] = \min[tz_1' + (1 - t) z_1'', tz_2' + (1 - t)z_2'', tz_3' + (1 - t)z_3''].$$

Let the $j$ th argument of the min function be its least element so

$$y[z't + z''(1 - t)] = tz_j' + (1 - t) z_j'' \geq t \min(z_1', z_2', z_3') + (1 - t) \min(z_1'', z_2'', z_3'')$$

$$= ty(z') + (1 - t) y(z'').$$

Thus, the function y is concave.

Next, use Theorem 2 of Rothschild and Stiglitz: If $H_1$ is more risky than $F_1$, then $E_{H_1} U(x) \leq E_{F_1} U(x)$ for every concave function U.

In particular, for every $z_2$ and $z_3$,

$$\phi_{H_1}(z_2, z_3) \equiv \int y(a_0 + a_1 x_1 + \varepsilon_1, z_2, z_3) dH_1 \leq \int y(a_0 + a x_1 + \varepsilon_1, z_2, z_3) dF_1 \equiv \phi_{F_1}(z_2, z_3).$$

Therefore,

$$Y_{H_1} = \int\int\int y \; dH_1 \, dF_2 \, dF_3 = \int\int \; \phi_{H_1}(b_0 + b_1 x_2 + \varepsilon_2, P + \varepsilon_3) \, dF_2 \, dF_3$$

$$\leq \int\int \phi_{F_1}(b_0 + b_1 x_2 + \varepsilon_2, P + \varepsilon_3) \, dF_2 \, dF_3 = Y_F .$$

So $Y_{H_1} \leq Y_{F_1}$, which was to be shown.

**Appendix II**

Within the class of distributions that

(1) are expandable in Taylor series about any of their points that has positive probability weight, with the radius of convergence of the Taylor series so large as to encompass all points with positive probability,[1] or

(2) represent nonstochastic variables, which is to say, $F(\varepsilon) = 0$ if $\varepsilon < 0$ and $F(\varepsilon) = 1$ otherwise,

a quadratic aggregate function implies

(1) an underlying rectangular distribution for one input,

(2) a nonstochastic distribution for one of the other inputs, and

(3) any distribution for the remaining input that leaves it nonbinding everywhere.

If all of $F_1$, $F_2$, and $F_3$ are nonrandom, then the aggregate function is LRP, which is certainly not quadratic. Since at least one of the variables must be truly random, let $F_1$ be the one with positive density on the narrowest range. That is, there are numbers $k_0$ and $k_1$, possibly $\pm$ infinity, so that $f_1(\varepsilon_1) = 0$ whenever $y - A = \varepsilon_1 < k_0$ or $y - A = \varepsilon_1 > k_1$. By assumption, $F_1$ is expandable in Taylor series on that range $k_0$ to $k_1$, so $f_1$, the density, exists and is also expandable in Taylor series.

The ranges over which $F_2$ and $F_3$ have positive density (and are expandable in Taylor series) are $[k_{02}, k_{12}]$ and $[k_{03}, k_{13}]$, respectively. These variables are nonstochastic if $k_{02} = k_{12}$ and $k_{03} = k_{13}$, respectively. Let $l_0 = \min(k_{02} + B, k_{03} + P)$ and $l_1 = \min(k_{12} + B, k_{13} + P)$. Define $G^* = F_2^*F_3^*$. Then $G^* = 1$ if $y < l_0$, $G^* = 0$ for $y > l_1$, and, for $l_0 \leq y \leq l_1$, $G^*$ can be expanded in a power series. Define z as a point ($l_0 < z < l_1$ and $k_0 < z < k_1$) around which expansions of both $G^*$ and $f_1$ can be made, and define $\tilde{y}$ as $y - z$. Then with $D_nG^*$ referring to the *n*th derivative of G, the Taylor series expansions for $G^*$ is

(1) $$G^* = \sum_{n=0}^{n=\infty} D_n G^*(z)\left(\frac{\tilde{y}^n}{n!}\right) \qquad \text{for } l_0 \leq y \leq l_1.$$

For $f_1$,

(2) $$f_1 = \begin{cases} 0 & \text{if } y - A < k_0, \\ \sum_{n=0}^{n=\infty} D_n f_1(z)\left(\frac{\tilde{y}^n}{n!}\right) & \text{if } k_0 \leq \varepsilon_1 = y - A \leq k_1, \\ 0 & \text{if } y - A > k_1. \end{cases}$$

The following shows that a quadratic aggregate function requires $l_0 = l_1 = 0$ and $F_1$ to be a rectangular distribution.

Proving this result requires consideration of all possible arrangements of $l_0$, $l_1$, $k_0$, and $k_1$, and examining what they imply about the marginal product of $x_1$, which is given by equation (13) in the text. Since the aggregate function is quadratic, the marginal products must be linear in $x_1$, $x_2$, and P; or, because A and B are linear in $x_1$ and $x_2$, it must be linear in A, B, and P. That is, $\partial Y/\partial A$ is

(3) $$\int f_1(y - A)\, G^*(y)dy = e_0 + e_1 A + e_2 B + e_3 P,$$

where the $e_i$s are constants.

The possible arrangements of the l's and k's are most usefully grouped by considering the interval in which both G* and $f_1$ are expandable in power series, $J = [m_0, m_1] = \{[k_0 + A, k_1 + A] \cap [l_0, l_1]\}$. There are five cases for J: (1) the null set; (2) the real line; (3) $[m_0, m_1] = [l_0, l_1]$ and $l_i \neq k_i + A$ nor does $l_0 = l_1$; (4) $[k_0 + A, m_1]$ or $[m_0, k_1 + A]$; and (5) the point $l_0 = l_1$. We now show that only the fifth case can result in a quadratic.

*Case 1.* When J is empty, it is easy to show that response is not quadratic. If $G^*$ is 0 between $k_0$ and $k_1$, then the marginal product is 0, which cannot come from a quadratic aggregate function. Otherwise, $G^* = 1$ between $k_0$ and $k_1$, while $f_1 = 0$ between $l_0$ and $l_1$.

$$(4) \qquad \int_{-\infty}^{\infty} f_1 G^* \, dy = \int_{k_0+A}^{k_1+A} f_1 \, dy$$

because $f_1$ is 0 elsewhere on the real line. This integral is 1 since $G^*$ is 1 over this range and f is a density with its integral 1. Since the integral is a constant and not linear, it cannot be the marginal product of a quadratic function. These are just the cases where one constraint binds with probability 1, resulting in a linear response to the binding input, and $\partial y/\partial x_1 = a_1 \int_{-\infty}^{\infty} f_1 G^* \, dy$ = either $a_1$ or 0. Thus, J empty gives a constant marginal product which violates the assumption of a quadratic total function.

*Case 2.* When J is the whole real line, both functions are everywhere expandable in Taylor series. Separate the Taylor series expansion of $f_1G^*$ around z into its first term and all other terms: $f_1(z)G^*(z) + \Theta(y - A, y - B, y - P)$. Here $\Theta$ is a polynomial in its arguments but does not have a constant term. It is just the terms (in powers of y - A, y - B, and y - P or higher) of the product of the Taylor expansions of $f_1$ and $G^*$ about z. Using this Taylor series expansion, the linearity of the marginal products requires

$$(5) \int_{-\infty}^{\infty} f_1(y - A) G^*(y) dy = f_1(z) G^*(z)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \Theta \, dy = e_0 + e_1 A + e_2 B + e_3 P.$$

A solution to this equation is impossible since integrating the constant $f_1(z)G^*(z)$ over the real line gives infinity, which, when added to the integral of $\Theta$, is still infinity and is certainly not equal to a linear function of A, B, and P.

*Case 3*. The case $J = [l_0, l_1]$ and not a point is just $A + k_0 < l_0 < l_1 < A + k_1$, which is impossible because it requires one of $F_2$ and $F_3$ to have nonzero density only on a range narrower than $f_1$, thus violating the assumption that input 1 had the narrowest distribution.

*Case 4*. When one or both of the boundaries of J is $A + k_i$, $\int_{-\infty}^{\infty} f_1 G^* \, dy$ can be broken into three integrals, one of which is of the form $\int_{m_0}^{m_1} f_1 G^* \, dy$, where either $m_0 = A + k_0$, $m_1 = A + k_1$, or both. Since, over this range, $f_1 \neq 0$ and $G^*$ takes values other than 0 or 1, $\partial Y/\partial A$ is

$$(6) \qquad \int_{m_0}^{m_1} f_1 G^* \, dy = \int_{m_0}^{m_1} [f_1(z) G^*(z) + \Theta] dy = (m_1 - m_0) f_1 G^* + \int_{m_0}^{m_1} \Theta(y) \, dy,$$

where $\Theta(y)$ is either 0 or a polynomial in y of degree 1 or greater. If it isn't zero, the integral of $\Theta(y)$ is of degree two or greater in $m_0$ and $m_1$. Since at least one of the m's is linear in A, $\int_{m_0}^{m_1} \Theta(y) dy$ would have to be at least quadratic in A. But $\int_{m_0}^{m_1} \Theta(y) dy$ quadratic in A violates the linearity of $\partial Y/\partial A$. Thus neither $f_1$ nor $G^*$ can have more than one nonzero term in the Taylor expansions, which is to say they must both be the constant functions. (Only one of the three integrals of y is presented here: the rest are either 0 or are integrals of $f_1$ over some range and do not change the conclusion.)

Thus, $f_1$ and $G^*$ must both be constants. By assumption, $G^*$ is neither 0 nor 1 over the range of this integral, so it is some other constant. Since the Taylor series expansion is valid for $[l_0, l_1]$, $G^*$ is constant on that interval as well.. If it is a constant other than 0 or 1 over this range, it must become one at $l_1$, the end of the range. Thus $G^*$ has a jump at $l_1$. It has probability mass at $l_1$, which is to say its density does not exist at that point. This contradicts the assumption that the distributions are expandable in power series (and therefore have densities) at every point at which they have probability mass. Therefore, this arrangement also fails to produce linear marginal products and thus a quadratic aggregate function.

*Case 5.* This leaves only the case $k_0 + A < l_0 = l_1 < k_1 + A$, precisely the case examined in detail in the text. In that case,

$$\int_{-\infty}^{\infty} f_1 G^* \, dy = \int_{k_0+A}^{l_0} f_1 \, dy = e_0 + e_1 A + e_2 B + e_3 P. \tag{7}$$

Note the definition of $l_0$ as $\min(k_{02} + B, k_{03} + P)$. When $l_0 = k_{02} + B$, the integral, (7), is linear in A and B only if $f_1$ is a constant. If $f_1$ is constant, then it is the density function for the rectangular distribution. Because $l_0 = l_1$, $\varepsilon_2$ must be deterministic. It is a point mass at zero. Since $k_{02} + B < k_{03} + P$, the plateau never binds and its distribution is irrelevant. If $l_0 = k_{03} + P$, then the plateau is deterministic and always binds before B. Thus, one input rectangular, one nonstochastic, and one irrelevant is the only case leading to a quadratic aggregate function. Since one of the inputs or the plateau is irrelevant (that is, one of these will never bind before the other), the aggregate function is quadratic in only two of the plateau and the two inputs. The remaining input or plateau does not enter the function.

## Footnotes

[1]The assumption on the radius of the Taylor series implies that the function does not have any poles, that is, any asymptotes, within that radius.

## Reference

Rothschild, Michael, and Joseph E. Stiglitz. "Increasing Risk: I. A Definition." *J. Econ. Theory* 2(1970):225-43.