Journal of Agricultural and Resource Economics, 17(2): 303–313

Existence of Unique Limiting Probability Vectors in Stochastic Processes with Multiple Transition Matrices

James W. Mjelde, Wesley D. Harris, J. Richard Conner, Gary D. Schnitkey, Michael K. Glover, and Lee Garoian

Concepts associated with stochastic processes containing multiple transition matrices are discussed. It is proved that under certain conditions, a process with m transition matrices has m unique limiting probability vectors. This result extends the notion of discrete Markov processes to problems with intrayear and interyear dynamics. An example using a large DP model illustrates the usefulness of the concepts developed to applied problems.

Key words: dynamic programming, limiting probability vectors, stochastic processes.

Introduction

Current trends in empirical research are to incorporate intrayear and/or interyear dynamics into modeling efforts (Antle; Mjelde, Dixon, and Sonka; Gustafson; Chiao and Gillingham). Along with incorporating dynamics, many studies also include the stochastic nature of the dynamic process. One application of stochastic process principles in these dynamic models has been to calculate the probability of being in a certain state of the process at different time periods. These probabilities have been (*a*) used to calculate expected long-run yearly net returns (Burt and Allison), (*b*) reported as a technique to summarize the dynamic nature of large dynamic programming models (Schnitkey, Taylor, and Barry; Mjelde, Taylor, and Cramer), and (*c*) reported to indicate the dynamics of an industry (Disney, Duffy, and Hardy). A shortcoming of these previous studies is the lack of rigorous treatment of stochastic processes involving multiple transition matrices. The objective of this study is to formalize such stochastic processes and prove the existence of unique limiting probability vectors associated with these processes.

Three studies that addressed processes with multiple transition matrices are Burt and Allison; Mjelde, Taylor, and Cramer; and Garoian, Mjelde, and Conner. None of these studies developed a methodology or theoretical basis applicable to the general case, although Garoian, Mjelde, and Conner suggested such a methodology is needed. In the current study, the existence of unique limiting probability vectors is proved for a general class of discrete stochastic processes with multiple transition matrices. The proof relies on previously developed theorems, which are not proved here. The discussion first develops stochastic processes with two transition matrices and then is expanded to m transition matrices. Using unique limiting probability vectors to summarize dynamic pro-

The authors are, respectively, associate professor, graduate research associate, and professor, Texas A&M University; assistant professor, The Ohio State University; and econometricians with American Express, TRS, Inc.

This is Texas Agricultural Experiment Station Technical Article No. 30238. This study was partially funded by Texas Agricultural Experiment Station Projects H-6507 and H-6193.

Comments by O. R. Burt, B. A. McCarl, D. A. Bessler, J. T. LaFrance, and anonymous *Journal* reviewers are gratefully acknowledged.

gramming solutions is one application of the techniques presented here. The relationship between converged optimal value functions, convergent decision rules, and unique limiting probability vectors associated with convergent decision rules is discussed. Finally, an example of a process with multiple transition matrices is given using a previously published calf/yearling marketing model (Garoian, Mjelde, and Conner). The example illustrates that these processes are relevant in empirical work and can be applied to large models.

Discrete Stochastic Processes with Two Transition Matrices

As noted earlier, current trends in empirical research are to include intrayear along with interyear dynamics. With such models, a single transition matrix will not suffice in specifying the model. In the case of a semiannual model, two unique transition probability matrices normally are specified, whereas a seasonal model would specify four transition matrices. Usually, in these types of models, the transition matrices vary by period within a year, but are identical between years. To clarify, consider a semiannual model. The transition matrix from January to July would be the same each year, but would differ from the July to January transition matrix. If the transition matrices did not differ, the model would not be a semiannual model.

Two basic elements necessary to formally define discrete stochastic processes with multiple transition matrices (DSPMTM) are the state of the system and state transitions. The state of the system describes the condition of the process at a particular stage or decision point. For example, in a semiannual model, each cycle of transitions would be a year with two stages during the year. State transitions describe changes in the state of the system from one period to the next.

Let $P1_{ij}$ and $P2_{ij}$ ($i, j = 1, 2, \ldots, k$ and $0 \leq P1_{ij}, P2_{ij} \leq 1$) represent the transition probabilities for a system with k possible states. The transition probability $P1_{ij}$ represents the probability of the system moving from state i to state j at the first transition of the process (e.g., January to July). Similarly, $P2_{ij}$ represents the probability of the system moving from state i to state j at the second transition of the process (e.g., July to January). These transitions are then repeated for each cycle (year) of the process life. Furthermore, the following conditions must hold:

$$(1) \qquad \sum_{j=1}^{k} P1_{ij} = 1 \qquad \text{for all } i, \quad \text{and}$$

$$\sum_{j=1}^{k} P2_{ij} = 1 \qquad \text{for all } i.$$

Equation (1) states that if the system is in state i, it must be in one of the k states after the next transition. Transition matrices, $P1$ and $P2$, are $k \times k$ matrices of all the transition probabilities, $P1_{ij}$ and $P2_{ij}$.

Let $\pi1_i(t)$ and $\pi2_i(t)$ be defined as the probabilities that the system will occupy state i at stages one and two during transition cycle t. It follows that

$$(2) \qquad \sum_{i=1}^{k} \pi1_i(t) = 1 \qquad \text{for all } t, \quad \text{and}$$

$$\sum_{i=1}^{k} \pi2_i(t) = 1 \qquad \text{for all } t.$$

Equation (2) forces the system to occupy some state defined in the system at each stage. Define $\pi1(t)$ and $\pi2(t)$ to be row vectors of the state occupancy probabilities containing elements $\pi1_i(t)$ and $\pi2_i(t)$. To find the probability that the system occupies some state at stage two, simply postmultiply the state occupancy probability vector at stage one by the transition matrix, $P1$. A similar procedure is used to find the state occupancy probability vector at stage one. That is,

$$
\begin{aligned}
(3) \qquad \pi 2(0) &= \pi 1(0)\, P1 \\
\pi 1(1) &= \pi 2(0)\, P2 = \pi 1(0)\, P1\; P2 \\
\pi 2(1) &= \pi 1(1)\, P1 = \pi 1(0)\, P1\; P2\; P1 \\
\pi 1(2) &= \pi 2(1)\, P2 = \pi 1(0)\, P1\; P2\; P1\; P2 = \pi 1(0)[P1\; P2]^2 \\
\pi 2(2) &= \pi 1(2)\, P1 = \pi 1(0)[P1\; P2]^2\, P1 \\
&\;\;\vdots \\
\pi 1(t) &= \pi 2(t-1)\, P2 = \pi 1(0)[P1\; P2]^t \\
\pi 2(t) &= \pi 1(t)\, P1 = \pi 1(0)[P1\; P2]^t P1,
\end{aligned}
$$

where $\pi 1(0)$ is the initial state occupancy vector.[1] For a process with m transition matrices, equation (3) easily can be modified to include m state occupancy vectors, $\pi 1(t)$ through $\pi m(t)$, and m transition matrices, $P1$ through Pm.

Unique limiting probability vectors would have the following properties:

$$(4) \qquad \pi 1 = \pi 2\; P2, \quad \text{and} \quad \pi 2 = \pi 1\; P1.$$

Equation (4) states that the probability of being in a given state at a given stage within a transition cycle does not change with an additional transition. That is, the probabilities for a given stage or transition are independent of t. Limiting probability vectors occur as t approaches positive infinity. When the limiting probability vector is unique, the initial probability vector, $\pi 1(0)$, does not influence the limiting probability vector. The process, therefore, loses its memory of its initial state provided enough transitions have occurred. The remainder of this article is devoted to the existence and use of unique limiting probability vectors.

Digression on Markov Processes

DSPMTM are very similar to discrete stochastic Markov processes. In fact, if $P1 = P2$, then the process becomes a discrete Markov process (only one unique transition matrix is present in the process). Because the proof presented below relies on the existence of unique limiting probability vectors for Markov processes, a short discussion of Markov processes and the existence of unique vectors is presented.

Under certain conditions, discrete Markov processes will possess a unique limiting probability vector. A limiting probability vector, π, has the following property:

$$(5) \qquad \pi = \pi\; P.$$

A sufficient condition for uniqueness of a limiting probability vector is that the transition matrix, P, be a regular matrix (Rorres and Anton). A regular transition matrix has the property that at least one integer power of the matrix has all positive elements. This condition implies that the process can go from any state to any other state, but not necessarily in one transition. A necessary condition for uniqueness is that the matrix, P, contain only one regular ergodic set (Howard; Kemeny and Snell). An ergodic set has a "closed" characteristic: Once the process enters the ergodic set, it can never leave this set, but the process can move between states within the set. The necessary condition implies that within the ergodic set, the process can go from any given state in the ergodic set to any other state within the ergodic set, but not necessarily in one transition. The remaining states are considered transient states which have a limiting probability of zero. Transient sets have the following characteristics: The system can move between the states in the transient set or the system can leave the transient set and enter another set. Once the process leaves a transient set of states, it can never return to this set. It should be noted that a regular matrix is just a special case of a matrix with a single ergodic set, the case in which no transient sets exist. For a more thorough discussion of these concepts, see Howard; Kemeny and Snell; or Mjelde et al.

Existence of Unique Limiting Probability Vectors in DSPMTM

The proof that unique limiting probability vectors exist in DSPMTM relies on the theorem that a transition matrix that contains a single ergodic set has a unique limiting probability vector. A proof of the uniqueness of limiting probability vectors with a transition matrix that contains a single ergodic set can be found in Kemeny and Snell or in Howard. First, the uniqueness of limiting probability vectors in DSPMTM is developed for two transition matrices. The theorem is then generalized to the case with m transition matrices.

Theorem: For processes defined in equation (3), unique limiting probability vectors exist for $\pi 1$ and $\pi 2$, if $[P1\ P2]$ is a single-set ergodic transition matrix.

Proof: To prove that unique limiting probability vectors exist, first consider the calculation of $\pi 1(t)$,

$$\pi 1(t) = \pi 1(0)[P1\ P2]^t. \tag{6}$$

It follows from the previous discussions that if the matrix resulting from the multiplication of $P1$ and $P2$ is a transition matrix that contains a single ergodic set, $\pi 1$ has a unique limiting probability vector as t goes to positive infinity. That is, when the matrix $[P1\ P2]$ is a transition matrix with a single ergodic set, $\pi 1$ has a unique limiting probability vector such that for a sufficiently large t the following condition holds:

$$\pi 1(t) = \pi 1(t - 1). \tag{7}$$

For a certain set of conditions, a unique limiting probability vector for $\pi 1$ therefore exists. Now consider $\pi 2$. For a unique limiting probability vector to exist for $\pi 2$, the following condition must hold for a sufficiently large t:

$$\pi 2(t) = \pi 2(t - 1). \tag{8}$$

The left and right side components of equation (8) can be calculated as

$$\pi 2(t - 1) = \pi 1(t - 1)\, P1, \tag{9}$$

and

$$\pi 2(t) = \pi 1(t)\, P1. \tag{10}$$

For a sufficiently large t, equation (7) holds. Combining equation (7) and equation (10) gives

$$\pi 2(t) = \pi 1(t)\, P1 = \pi 1(t - 1)\, P1. \tag{11}$$

Equation (9), however, states

$$\pi 2(t - 1) = \pi 1(t - 1)\, P1; \tag{12}$$

therefore,

$$\pi 2(t) = \pi 2(t - 1), \tag{13}$$

which is a necessary condition for a unique limiting probability vector for $\pi 2$ to exist. Further, if $\pi 1$ is unique, then $\pi 2$ will be unique because equation (12) reduces to a unique vector, $\pi 1$, postmultiplied by a given matrix, $P1$; therefore, $\pi 2$ must be unique.

The theorem relies on the assumption that $[P1\ P2]$ is a transition matrix with a single ergodic set. Mjelde et al. showed that the multiplication of two transition matrices results in a transition matrix. Intuitively, this component of the assumption must hold or equation (3) with the matrix raised to a power would not hold. Assuming that $[P1\ P2]$ is a transition matrix therefore is not restrictive, but simply follows from multiplication of two transition matrices.

The theorem then rests on the assumption that [$P1$ $P2$] has a single ergodic set. A sufficient condition for [$P1$ $P2$] to contain a single ergodic set is that either $P1$ or $P2$ contains a single ergodic (see Mjelde et al. for a proof). This sufficient condition, however, is not a necessary condition. As illustrated in Mjelde et al., both the $P1$ and $P2$ matrices can include many ergodic sets and the [$P1$ $P2$] matrix may contain only one ergodic set. In these cases, a necessary condition for the [$P1$ $P2$] matrix to contain only one ergodic set is that the sets in stage one's transition matrix be overlapped by a set (either transient or ergodic set) in stage two's matrix. To illustrate, suppose $P1$ has an ergodic set consisting of states 1 and 2 and another ergodic set containing states 3 and 4. The matrix [$P1$ $P2$] will be single-set ergodic only if $P2$ contains a set (transient or ergodic) that has at a minimum state 1 or 2 and state 3 or 4.

Unique limiting probability vectors, therefore, may exist even if all the intrayear matrices contain more than one ergodic set. This illustrates the need to consider the [$P1$ $P2$] matrix and not the $P1$ and $P2$ matrices individually. It also explains why the assumption of [$P1$ $P2$] containing a single ergodic set is necessary. Determining the status of the [$P1$ $P2$] matrix can be accomplished using the same methods applicable to all transition matrices. Methods for examining matrices are presented in Mjelde et al. and in Heyman and Sobel.

The status of sets in the [$P1$ $P2$] matrix also holds implications for the status of the [$P2$ $P1$] matrix. If the $\pi 1$ vector is unique, then the $\pi 2$ vector also is unique, thereby implying that the [$P2$ $P1$] matrix is single-set ergodic. This implication occurs because a necessary condition for unique limiting probability vectors is that the transition matrix contain only one ergodic set. Conversely, if $\pi 1$ is not unique, which results because [$P1$ $P2$] is not single-set ergodic, then $\pi 2$ also is not unique (if $\pi 2$ is unique, then $\pi 1$ would also be unique by modification of the previous proof). This implies that the [$P2$ $P1$] matrix contains more than one ergodic set. Either [$P1$ $P2$] and [$P2$ $P1$], therefore, are both single-set ergodic or they both contain more than one ergodic set. This allows examination of only one of the postmultiplied matrices when determining the status of all the possible combinations of postmultiplied intrayear matrices.

Generalization to *m* Transition Matrices

Rewriting equation (3) to represent m transition matrices instead of two transition matrices results in

$$\pi 1(t) = \pi m(t-1)\, Pm = \pi 1(0)\, [P1\ P2 \ldots Pm]^t, \tag{14}$$

and for $n \neq 1$,

$$\pi n(t) = \pi(n-1)(t) P(n-1) = \pi 1(0)\, [P1\ P2 \ldots Pm]^t\, P1\ P2 \ldots P(n-1), \tag{15}$$

where $n = 2, \ldots, m$ and the remaining variables are as defined in equation (3) with appropriate modification for greater than two transition matrices. From equations (14) and (15) it can be seen that the arguments presented in the two transition matrix process case apply with the modification that [$P1$ $P2 \ldots Pm$] contains a single ergodic set. A process with m transition matrices, therefore, will have m unique limiting probability vectors when the matrix [$P1$ $P2 \ldots Pm$] contains a single ergodic set.

Be it a process with one transition matrix or m transition matrices, the relevant consideration is the entire process and not the individual transition matrices. This is embedded in the necessary assumption for the existence of unique limiting probabilities, namely that [$P1$ $P2 \ldots Pm$] contains a single ergodic set. Another way of interpreting the necessary assumption is that [$P1$ $P2 \ldots Pm$] gives a Markov process transition matrix covering the entire time period for the original DSPMTM.

Relationship to Dynamic Programming

As noted earlier, limiting probability vectors have been used in previous Markov processes studies. In these studies, only interyear dynamics were formalized. This study formalizes the use of limiting probability vectors to problems exhibiting both intra- and interyear dynamics. The most obvious use of limiting probability vectors is the simple application of equations (14) and (15). A second use is to summarize dynamic programming (DP) results. It is this use of limiting probability vectors that comprises the remainder of this study.[2]

Usually, decision rules and optimal value functions from DP exhibit convergence. Considerable literature exists concerning necessary and sufficient conditions for the existence of converged decision rules and optimal value functions (Bellman; Bertsekas 1974, 1976; Blackwell; Dreyfus and Law). These studies focus on factors such as the discount rate, discreteness, and finiteness of the state space, and transforming a nonstationary system to a stationary system. At first glance, it would appear that the existence of unique decision rules and/or converged optimal value functions would imply the existence of a unique limiting probability vector associated with a transition matrix from the convergent decision rule.[3] This, however, is not the case. A converged decision rule does not guarantee the existence of a single-set ergodic transition matrix. The following example illustrates that these are distinct concepts. However, it remains true that if the *full* transition matrix converges uniquely and the immediate net returns do not vary by stage, then the decision rule also must converge in all cases of practical interest.

To show that the existence of a convergent decision rule and a converged optimal value function does not imply the probability vector associated with the convergent decision is unique, consider the following Markov problem. The problem has four possible states and two possible decisions at each stage. The probability of being in a given state at stage $t + 1$ depends on the state at time t and the decision undertaken. Let the following matrices represent the transition probabilities associated with each decision:

$$P_{ij}^{1} = \begin{bmatrix} .2 & .2 & .2 & .4 \\ .8 & .2 & 0 & 0 \\ .1 & .9 & 0 & 0 \\ .5 & .5 & 0 & 0 \end{bmatrix} \tag{16}$$

and

$$P_{ij}^{2} = \begin{bmatrix} 0 & 0 & .5 & .5 \\ 0 & 0 & .2 & .8 \\ .1 & .5 & .2 & .2 \\ .2 & .2 & .4 & .2 \end{bmatrix}. \tag{17}$$

These matrices correspond to the previous definition of transition matrices. In these matrices, the rows indicate the state in stage t and the columns indicate the state in stage $t + 1$. For example, the third row in the first matrix indicates that if you are in state 3 at stage t and undertake decision 1, the probability at stage $t + 1$ of being in state 1 is .1, state 2 is .9, and zero for the remaining states. Let the immediate net returns for the problem be a function of the decision and state, that is,

State	Decision 1	Decision 2
1	0	20
2	0	20
3	20	0
4	20	0

The objective is to maximize expected net returns results in the following recursive equation:

(18) $$V_t(i) = \max_D\{\text{ret}(i, D) + \beta \sum_{j=1}^{4} P_{ij}^D V_{t+1}(j)\},$$

where $V_t(i)$ is the optimal value from t to the end of the time horizon given the process is in state i and the optimal policy is followed, ret(i, D) is the immediate net return function, β is the discount factor, and P_{ij}^D is the probability of going from state i at stage t to state j at stage $t + 1$ given decision, D. For any positive discount rate, repeated application of the recursive equation results in the following optimal decision rule: If the process is in states 1 or 2 the optimal decision is 2 and if the process is in states 3 or 4 the optimal decision is decision 1. The example, therefore, has a convergent decision rule. Further, for any positive discount rate, the example's optimal value function satisfies the conditions necessary to converge.

To calculate limiting probability vectors based on the convergent decision rule, the following transition matrix is developed. This transition matrix is comprised of the first two rows of the decision 1 matrix and the last two rows of the decision 2 matrix, that is,

(19) $$\begin{bmatrix} 0 & 0 & .5 & .5 \\ 0 & 0 & .2 & .8 \\ .1 & .9 & 0 & 0 \\ .5 & .5 & 0 & 0 \end{bmatrix}.$$

Repeated application of equation (5), given an initial state occupancy vector of 1 for states 1 or 2, yields the following state occupancy vectors:

(20) $$\pi(t) = [\,0 \quad 0 \quad .31 \;\; .69] \quad \text{if } t \text{ is odd and}$$
$$\pi(t) = [.38 \;\; .62 \quad 0 \quad 0\,] \quad \text{if } t \text{ is even.}$$

Obviously, $\pi(t)$ does not equal $\pi(t + 1)$; therefore, a unique limiting probability vector does not exist. This occurs because the transition probability matrix (19) is cyclical. Kemeny and Snell showed that cyclical matrices do not have unique limiting probability vectors. Neither a convergent decision rule nor a converged optimal value function (nor both together), therefore, implies that a unique limiting probability vector associated with the convergent decision rule exists.

When using the techniques discussed here to summarize DP decision rules, the above example illustrates the necessity to examine the matrix associated with the convergent decision rules to determine if it is single-set ergodic. This occurs because the calculations of limiting probability vectors based on convergent decisions are separate from the actual DP model. In fact, any decision rule (optimal, converged, or nonoptimal) can be used in the techniques described here. Further, the transition matrices associated with the other decision rules must be examined to determine if they are single-set ergodic. The discussion presented here could be expanded easily to DSPMTM processes. Examination of the transition matrices, therefore, is important in both Markov and DSPMTM processes.

Application to Determine Expected Long-Run Yearly Net Returns

Burt and Allison illustrated that the calculation of expected long-run returns can be used to compare models or decision rules. Expected long-run returns are calculated by multiplying the limiting probability vector by a vector of yearly net returns. For the m

transition matrix case, expected yearly long-run net returns, $E(LRNR)$, are calculated by summing the products from the multiplication of the limiting probability vectors and net return vectors, that is,

$$E(LRNR) = \sum_{i=1}^{m} \pi i R i, \tag{21}$$

where Ri represents a vector of net returns associated with each time period and state. In this section, we apply equation (21) to a previously published model to examine optimal cow herd size.

Problem Statement

Garoian, Mjelde, and Conner developed a DP marketing model for rangeland calves and yearlings. They developed optimal marketing strategies for several herd sizes and expected net returns from following the optimal decision rule over a 10-year planning horizon. Within their model, calves can be sold at weaning (October), as short yearlings (May), or as long yearlings (following October). The convergent optimal decision rules were relatively robust with respect to the number of brood cows. Disregarding price effects, the optimal decisions were to keep all calves in October that could be supported by the standing crop level without supplemental feeding. As prices increased, more calves were sold to take advantage of the higher fall price. As herd sizes increase, more calves are sold at lower prices than when herd sizes are smaller. This is because of the higher forage requirements of the larger herds and potentially higher supplemental feeding costs. In May, short yearlings are retained only under a very limited state space under the smaller herd sizes. These results indicate that it is not profitable for a cow-calf operation to retain calves or yearlings and feed them in either October or May. See Garoian, Mjelde, and Conner for a more thorough discussion of the convergent decision rules, especially the 275-cow herd size.

Within the model discussed above, cow herd size is an exogenous variable; therefore, optimal herd size cannot be determined within the model. With regard to the net returns, the results showed that initial conditions determined which herd size outperformed the other herd sizes over a 10-year planning horizon. That is, no one herd size dominated the other herd sizes for all possible initial conditions. Using limiting state occupancy probabilities, expected long-run yearly net returns for various herd sizes are calculated in the following section to determine which herd size, on average, would return the most to the rancher.

Calf/Yearling Marketing Model

Two stochastic state variables, cattle price and standing crop (amount of available forage), and a deterministic state variable, inventory of yearlings on hand, are specified within the marketing model. Cattle prices are represented by 11 levels, standing crop by 8 levels, and inventory by 11 levels. Therefore, 968 (11 × 8 × 11) possible states are defined within the model at any given time period. Standing crop ranges from zero to 2,856 lbs./acre, price ranges between \$30/cwt and \$85/cwt, and inventory ranges between zero and 100% of the calf crop. Both the standing crop and price levels are represented by equally spaced intervals, whereas the inventory state variable represents increments of 10% of the weaned calf crop.

Two decision points or stages, May and October, are considered. In October the rancher has the option of either selling the current year's calf crop or retaining the calves to be sold the next year either in May or October. In May the rancher can either sell the calves as short yearlings or retain them until October when they must be sold as long yearlings (calves from last year's crop). The decision at both May and October is the number of calves to sell; this decision is represented by 11 levels. The first level represents retaining ownership of all calves or yearlings. The next 10 levels represent selling 10%, 20%, 30%, etc., up to 100% of the weaned calf crop. Calving rate is assumed to be 80% of the cow herd size.

Within the model, various components make up the immediate net returns for each stage. These components vary by stage, decision undertaken, and the state of the process. Returns from the sale of calves or yearlings is the most obvious component. A second component is variable costs associated with the cow-calf operation. Each October a per cow variable cost of $114.83 is imposed on the net return function. A per yearling retained cost of $27.05 is imposed in May and $48.26 in October. The last major component is supplemental feed costs. When livestock forage demands are greater than the available standing crop, the cost of purchasing and feeding range cubes is added to the net returns function. This cost depends on the number of cows, calves, and yearlings present on the range. The decision to sell or retain calves, therefore, directly affects the supplemental feeding costs. No fixed costs associated with land and equipment are included in the model.

The objective of the model is to maximize the expected net returns associated with the cow-calf-yearling operation. A rancher may retain calves in October to sell as short (May) or long (October) yearlings due to price fluctuations. Typically, in the fall calf prices are at their lowest because the supply of calves is at its highest. Profits may be increased if the calves are retained and sold as short or long yearlings. But, typically, the price of heavy-weight yearlings is discounted relative to lower-weight animals. Capturing seasonal price fluctuations could offset this loss in price. Decision rules discussed previously indicate that retaining calves in the fall and selling as short or long yearlings is profitable as long as forage is not limiting. Garoian, Mjelde, and Conner found that the increase in net returns from adopting a flexible marketing strategy comes from a more efficient utilization of the forage resource.

Application of Multiple Limiting Probability Vectors

Using the methodology discussed previously concerning limiting state occupancy vectors and equation (21), expected long-run yearly net returns and expected number of calves or yearlings retained are calculated. The transition probability matrices, which vary by time period, are based on convergent decision rules from the DP model. Convergent decision rules have the property that the decisions for a given stage (May or October) do not vary between years for a given state of the process. The rules between stages May or October can and do vary for any given state. Once the decision rules converge, two transition matrices (October to May and May to October) along with two net return vectors are calculated from the DP model. Equation (3) is then applied until limiting probability vectors are obtained. Finally, equation (21) is used to calculate net returns. The same procedure is used to calculate expected number of calves or yearlings retained.

Results

Expected long-run net returns and retained calves are calculated for four herd sizes (250, 275, 300, and 325 brood cows) to ascertain the possible "optimal" herd size for the rangeland scenario modeled in Garoian, Mjelde, and Conner. Expected net returns and calves/yearlings retained by stage are given in table 1. Total expected yearly net returns also are listed. In each of the four herd sizes, the combined transition matrix $[P1\ P2]$ contained a single ergodic set.

Differences in the expected net returns and the number of yearlings retained are shown in table 1. October net returns are negative for the 250 and 275 brood cow herd sizes. This is attributable to retaining the majority of the current year's calves and the variable costs associated with the cow herd being assessed in October. Expected net returns vary by stage for the four herd sizes, but for the smallest three herd sizes, expected yearly net

Table 1. Expected Net Returns and Expected Calves or Yearlings Retained by Stage

Herd Size	Total Calves	Expected Yearly Net Returns	October Net Returns	October Calves Retained	May Net Returns	May Calves Retained
		($)	($)		($)	
250	200	38,717	−2,858	124	41,575	9
275	220	39,535	−167	117	39,702	2
300	240	38,514	12,383	90	26,131	0
325	260	35,821	26,646	56	9,175	0

Note: Dollar amounts and retained calves/yearlings are rounded to the nearest whole number.

returns vary by only approximately $1,000. A cow herd size of 275 cows gives the highest expected yearly net returns. Expected yearly net returns for the 325-cow herd size are approximately $4,000 less than the 275-cow herd size. Although the net returns for the three smallest herd sizes are similar, the cash flows associated with the 300-cow herd size differ from the cash flows associated with either the 250- or 275-cow herd size. Cash flows associated with the 325-cow herd size again vary from the remaining three herd sizes.

In all herd sizes, some calves are expected to be retained in October and sold as either short or long yearlings. As expected, the number decreases as the size of the cow herd increases. This reflects increased forage demands of the larger cow herd sizes. Some short yearlings are expected to be retained in May to be sold as long yearlings in October for all herd sizes except 325 cows. Again, this reflects forage demands and price fluctuations.

Conclusions

Concepts associated with stochastic processes that possess multiple transition matrices have been discussed. Such processes account for seasonality or cyclical nature with a wide variety of real world problems. It was shown that under the condition that the transition matrix for the entire process contains only a single ergodic set, then unique limiting probability vectors exist. This result extends the notion of discrete Markov process principles to situations with intrayear and interyear dynamics, creating a more realistic situation for many types of processes.

An example of expected yearly net returns obtained from a DP model shows that the methodology developed in this article is applicable to large empirical problems. The example contained two transition matrices, each representing 968 distinct states. With today's increasing computer capabilities, larger-size problems easily can be handled. Summarizing DP solutions is only one application of the concepts discussed. Many other types of applications of limiting probability vectors are possible. Mjelde et al. discuss the concepts introduced in this study in greater detail, presenting several examples.

[*Received August 1991; final revision received March 1992.*]

Notes

[1] As noted by a reviewer, if the model is formulated with intrayear periods treated as simply another state variable, the two-period model would be characterized by the transition matrices:

$$P = \begin{bmatrix} 0 & P1 \\ P2 & 0 \end{bmatrix} \text{ and}$$

$$P^2 = \begin{bmatrix} P1\,P2 & 0 \\ 0 & P2\,P1 \end{bmatrix}.$$

In this case, it would not be necessary to analyze the high dimension matrix *P*. Analyzing [*P*1 *P*2] or [*P*2 *P*1] would suffice.

[2] We would like to thank the editor and reviewers for suggesting that this section be included.

[3] An optimal convergent decision rule is a rule which is stage independent; that is, for a given state the optimal decision rule does not vary by stage. The rule usually varies by state.

References

Antle, J. M. "Sequential Decision Making in Production Models." *Amer. J. Agr. Econ.* 65(1983):282–90.

Bellman, R. E. *Dynamic Programming.* Princeton NJ: Princeton University Press, 1957.

Bertsekas, D. P. *Dynamic Programming and Stochastic Control.* New York: Academic Press, 1976.

———. "Necessary and Sufficient Conditions for Existence of an Optimal Portfolio." *J. Econ. Theory* 8(1974): 235–47.

Blackwell, D. "Discounted Dynamic Programming." *Ann. Math. Statis.* 36(1965):226–35.

Burt, O. R., and J. R. Allison. "Farm Management Decisions with Dynamic Programming." *J. Farm Econ.* 45(1963):121–36.

Chiao, Y., and A. Gillingham. "The Value of Stabilizing Fertilizer Under Carry-over Conditions." *Amer. J. Agr. Econ.* 71(1989):353–62.

Disney, W. T., P. A. Duffy, and W. E. Hardy, Jr. "A Markov Chain Analysis of Pork Farm Size Distributions in the South." *S. J. Agr. Econ.* 20(1988):57–64.

Dreyfus, S. E., and A. M. Law. *The Art and Theory of Dynamic Programming.* New York: Academic Press, 1977.

Garoian, L., J. W. Mjelde, and J. R. Conner. "Optimal Strategies for Marketing Calves and Yearlings from Rangeland." *Amer. J. Agr. Econ.* 72(1990):604–13.

Gustafson, C. R. "Stochastic Dynamic Modeling: An Aid to Agricultural Lender Decision Making." *West. J. Agr. Econ.* 14(1989):157–65.

Heyman, D. P., and M. J. Sobel. *Stochastic Models in Operations Research,* Vol. 1: *Stochastic Processes and Operating Characteristics.* New York: McGraw-Hill, Inc., 1982.

Howard, R. A. *Dynamic Programming and Markov Processes.* Cambridge MA: The MIT Press, 1960.

Kemeny, J. G., and J. L. Snell. *Finite Markov Chains.* Princeton NJ: D. von Nostrand Company, Inc., 1960.

Mjelde, J. W., B. L. Dixon, and S. T. Sonka. "Estimating the Value of Sequential Updating Solutions for Intrayear Crop Management." *West. J. Agr. Econ.* 14(1989):1–8.

Mjelde, J. W., W. D. Harris, G. D. Schnitkey, M. K. Glover, L. Garoian, and J. R. Conner. "Application of Stochastic Processes to Summarize Dynamic Programming Solutions." Tech. Rep. No. DTR 91-1, Dept. Agr. Econ., Texas A&M University, 1991.

Mjelde, J. W., C. R. Taylor, and G. L. Cramer. "Optimal Marketing Strategies for Wheat and Corn Producers in the 1982 Farm Program." *N. Cent. J. Agr. Econ.* 7(1985):51–60.

Rorres, C., and H. Anton. *Applications of Linear Algebra.* New York: John Wiley & Sons, 1984.

Schnitkey, G. D., C. R. Taylor, and P. J. Barry. "Evaluating Farmland Investments Considering Dynamic Stochastic Returns and Farmland Prices." *West. J. Agr. Econ.* 14(1989):143–56.