DYNAMIC PROGRAMMING, ACTIVITY ANALYSIS AND THE THEORY OF THE FIRM

C. D. Throsby*

The role of dynamic programming as a means of examining the allocation and pricing problems in the theory of the firm is considered in this paper. The production relationships and equilibrium conditions as specified by neoclassical theory and linear programming are stated and dynamic programming formulations of each of these models are constructed and compared. It is demonstrated that dynamic programming adds nothing to established theory in these cases, providing simply an alternative means of computation which might be preferred for some empirical problems. It is concluded that some theoretical contribution may be possible by using dynamic programming to attack problems beyond the scope of conventional methods.

I INTRODUCTION

The theoretical implications of major innovations in quantitative economics have on occasion not been fully realized until some time after the initial contribution was made. Linear programming provides an example. The empirical application of this technique to quantitative optimization problems grew more spectacularly than did the realization of its implications, say, for the theory of the firm or for the theory of general equilibrium. If we take the initial "discovery" of linear programming as being in 1947, the date of Dantzig's original paper[1], it is not until five years or so afterwards that we see its ramifications into these two theoretical areas receiving substantial attention in the literature.[2]

* Supervisor of Research in Economics, Department of Agriculture, Sydney.

[1] G. B. Dantzig, "Maximization of a Linear Function of Variables Subject to Linear Inequalities", republished in T. C. Koopmans, *ed., Activity Analysis of Production and Allocation*, (N.Y.: John Wiley, 1951), pp. 339–47.

[2] For the two fields cited the earliest major theoretical landmarks would doubtless be R. Dorfman, *Application of Linear Programming to the Theory of the Firm*, (Berkeley: University of California Press, 1951), and K. J. Arrow and G. Debreu, "Existence of an Equilibrium for a Competitive Economy", *Econometrica*, Volume 22, No. 3, (July, 1954), pp. 265–90.

What of dynamic programming[3], which has now been a recognizable area of endeavour for about 15 years? Understandably its early days were taken up with methodological and empirical exploration during which time a remarkable variety of optimization problems proved soluble using its basic principles. But in the jungle of economic theory, dynamic programming has made little headway. Whereas linear programming has provided a means for constructing a model of the firm which is in many respects more acceptable than the neoclassical model, dynamic programming has enabled no such contribution. This is because ultimately dynamic programming reduces simply to a methodological trick, a means of maximization or minimization, and as such should more properly be compared with the calculus than with linear programming. Its chief contribution to theory has been that it has enabled the formulation and analysis of some problems which were previously either very cumbersome or downright impossible to solve. Such contribution has been made largely in terms of specific problems and few, if any, general theoretical conclusions have emerged.

This paper deals with allocation models, a field in which dynamic programming's empirical impact has been severely limited by the so-called "dimensionality problem".[4] Technological progress in the computer industry notwithstanding, this methodological affliction will doubtless continue to constrain development in empirical applications for some time to come. It seems useful at this stage, then, to consolidate one area of theory by considering the theoretical relations between dynamic programming, linear programming and neoclassical analysis in the theory of the firm.

2 ANALYSIS

For expositional convenience we shall consider a firm producing two products and using two factors. Let

b_i = total quantity of i-th factor available to the firm, $(i = 1, 2)$;

y_j = total output of j-th product $(j = 1, 2)$;

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[3] As defined and developed by Bellman. See R. E. Bellman and S. E. Dreyfus, *Applied Dynamic Programming* (Princeton: Princeton University Press, 1962).

[4] This refers to the fact that in a multi-input multi-output dynamic programming allocation problem, computational burden (in terms of both calculation time and computer memory requirement) increases linearly with the number of outputs considered, but exponentially with the number of inputs. Rearrangement of the formulation such that computation increases linearly with the number of inputs causes exponential calculation increases with the number of outputs. You can't have it both ways and hence any empirical model is restricted to consideration of only, say, two or three inputs or outputs. This is regrettable, since almost *any* allocation problem can be drawn up on paper in dynamic programming terms, and any number of refinements can be written into the set of equations. You want non-linear functions with some integer variables? Of course. Parameterization of prices and resource supplies? Easy. Replace some parameters with random variables? Just a stroke of the pen. It's unfortunate that the resulting set of equations would take a computer several decades to solve.

x_{ij} = total quantity of i-th factor allocated to j-th product;
p_j = price of the j-th product;
q_i = price of the i-th factor;
w_j = production function with respect to the j-th output.

The theory of production consists of a set of production relationships and a set of equilibrium conditions based on the supposition of profit maximization on the part of the entrepreneur. Let us begin by stating these briefly for the traditional and linear models.

NEOCLASSICAL MODEL

The basic production relationship is the production function, written down in general terms as:

(1)
$$y_1 = w_1(x_{11}, x_{21})$$
$$y_2 = w_2(x_{12}, x_{22})$$

For the purposes of deriving the equilibrium conditions a requirement of this model is that the functions involved be continuous with continuous first and second derivatives. A number of mathematical forms for the production relationships satisfy this requirement.

First-order equilibrium conditions for this model are that the marginal value product of any factor should be the same in each output and should in turn equal the factor price. (We take the second-order conditions for granted.) This set of conditions[5] may be written simply as follows:

(2)
$$\frac{\delta w_1}{\delta x_{11}} = \frac{\delta w_2}{\delta x_{12}} = q_1$$
$$\frac{\delta w_1}{\delta x_{21}} = \frac{\delta w_2}{\delta x_{22}} = q_2$$

In this model the resource availabilities, b_i, do not enter explicitly, as unlimited supplies of any factor are assumed available at the going price. The brake on expansion of output in this model is applied by characteristics of production reflected in (1), specifically diminishing marginal product.

LINEAR MODEL

Again we begin with the fundamental production relation shown in (1). In the case of the linear model, however, the functions w have a different form. To exhibit this, divide the first equation of (1) by y_1 and the second by y_2. Letting $\frac{x}{y} = a$, this yields:

[5] The statement of the equilibrium conditions as in (2) *implies*, of course, the alternative statements of these conditions, e.g., that the marginal rate of substitution between any pair of inputs should equal the inverse of their price ratio; etc.

$$(3) \qquad 1 = w_1\left(\frac{x_{11}}{y_1}, \frac{x_{21}}{y_1}\right) = w_1(a_{11}, a_{21})$$

$$1 = w_2\left(\frac{x_{12}}{y_2}, \frac{x_{22}}{y_2}\right) = w_2(a_{12}, a_{22})$$

The a's are the familiar linear programming input-output coefficients[6], and are constant in this model (whereas if derived from a continuous function as treated in the previous section, the a's would be continuous variates).

Assuming that an optimal solution exists, and omitting slack variables for simplicity, we may derive a set of equilibrium conditions for this model which define optimal non-negative values for the y's and q's. It is required that:

(a) total resource use exhausts the supply; i.e.,

$$(4) \qquad \begin{aligned} b_1 &= a_{11}y_1 + a_{12}y_2 \\ b_2 &= a_{21}y_1 + a_{22}y_2 \end{aligned}$$

(b) product price equals unit cost in each output; i.e.,

$$(5) \qquad \begin{aligned} p_1 &= a_{11}q_1 + a_{21}q_2 \\ p_2 &= a_{12}q_1 + a_{22}q_2 \end{aligned}$$

(c) profits are entirely distributed to factors of production; i.e.,

$$(6) \qquad p_1y_1 + p_2y_2 = z = b_1q_1 + b_2q_2$$

It is easily shown that these equilibrium conditions are equivalent to those of the "smooth model" stated above.[7] The difference between the two models is in their representation of production conditions. In the linear model expansion of output is constrained by limitations on the supplies of factors, rather than by diminishing returns in the production function.

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[6] A formulation perhaps more familiar in agricultural applications of linear programming is one in which a particular *resource* (say the k-th) is chosen as numeraire. Equation (1) is then divided through by x_k for each j. For example, taking the first resource in (1) as numeraire, we obtain:

$$(3a) \qquad \frac{y_1}{x_{11}} = w_1\left(1, \frac{x_{21}}{x_{11}}\right)$$

$$\frac{y_2}{x_{12}} = w_2\left(1, \frac{x_{22}}{x_{12}}\right)$$

If, as is frequently the case in farm planning models, the resource chosen as numeraire is land, the quantity $\frac{y_j}{x_{kj}}$ measures the output of product j per acre, and $\frac{x_{ij}}{x_{kj}} = a_{ij} =$ requirement of the i-th resource per acre of activity j.

[7] See R. Dorfman, P. A. Samuelson and R. M. Solow, *Linear Programming and Economic Analysis*, (N.Y.: McGraw-Hill, 1958), pp. 375–81. It should perhaps be pointed out that in this model the q's are *imputed* factor prices. Because supplies of factors are already possessed by the firm and fixed in the short-run, the market prices need not enter this model. Whilst the cash prices of factors in the real world do not necessarily equal their marginal value products in the production sector as a whole, they will do in long-run equilibrium under assumptions of perfect competition in the respective factor markets.

DYNAMIC PROGRAMMING

A dynamic programming representation of the firm problem under study may begin with the set of production relations in (1). Let us write down first a general model for the dynamic programming solution of the two factor allocation problem. The following recurrence relation, applied sequentially to all activities, yields optimal allocations of each resource to each activity as a function of resource availabilities[8]:

(7)
$$f_j(b_1, b_2) = \max_{\substack{0 \leqslant x_{1j} \leqslant b_1 \\ 0 \leqslant x_{2j} \leqslant b_2}} \{g_j(x_{1j}, x_{2j}) + f_{j+1}[(b_1 - x_{1j}), (b_2 - x_{2j})]\}$$

where $f_j(b_1, b_2)$ = returns from following an optimal policy over activities $j, j + 1, \ldots,$ for given amounts b_1 and b_2 of resources initially available;

$g_j(x_{1j}, x_{2j})$ = return function showing the profit earned by an allocation of x_{1j} units of resource 1 and x_{2j} units of resource 2 to activity j.

As before, in our present problem we take the simple case of $j = 1, 2$.

To demonstrate the relationship between this model and the two considered above we may show show how the return function g may be derived from a neoclassical production function or from the production function system embodied in the linear model. We may then consider the existence of equilibrium in these two cases and proceed to a more general specification.

(*a*) Take a Cobb-Douglas function as an appropriate first illustration. Letting α and β represent the usual parameters we have:

(8)
$$y_1 = \alpha_1 x_{11}^{\beta_{11}} x_{21}^{\beta_{21}}$$
$$y_2 = \alpha_2 x_{12}^{\beta_{12}} x_{22}^{\beta_{22}}$$

whence the dynamic programming return functions are derived simply as:

(9)
$$g_1(x_{11}, x_{21}) = p_1 \alpha_1 x_{11}^{\beta_{11}} x_{21}^{\beta_{21}} - q_1 x_{11} - q_2 x_{21}$$
$$g_2(x_{12}, x_{22}) = p_2 \alpha_2 x_{12}^{\beta_{12}} x_{22}^{\beta_{22}} - q_1 x_{12} - q_2 x_{22}$$

Computation proceeds with (7) yielding a set of optimal allocations of the two factors to the two products as a function of amounts of factors available. Contrary to first appearances the computation does not require an assumption that factor supplies are limited; rather b_1 and b_2 can be specified initially large enough to ensure that the optimal unlimited-factor-supply solution is obtained. To explain this further, let us suppose

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[8] The dynamic programming methodology is explained in Bellman and Dreyfus, *op. cit.*, esp. Chs I and II. The "one-stage" problem, frequently written as a separate equation to (7), is of course just a special case of (7) for which the function f_{j+1} does not exist.

that for a given problem the optimal allocation of factor i to output j is x^*_{ij}. Now consider input i. Since an optimal solution in this model is dictated for given values of all other parameters by the diminishing marginal product characteristics of the production function, the optimum will not alter for factor supplies of input i in excess of $\sum_j x^*_{ij}$. Hence in arbitrarily choosing b_i initially, it is necessary only to ensure that it exceeds this level, a process achievable at best by judicious guesswork, at worst by trial and error.

Once the optimal input levels have been obtained, the corresponding levels of output are determined by substitution into (8).

(*b*) Turning to the linear model we may derive return functions from (3). For any given allocation of resources to activity, j, output can expand only as far as the constraint imposed by the most limiting resource at that allocation. Mathematically this is expressible in our notation, for any matrix of allocations x_{ij}, as the set of equations

$$y_j = \operatorname*{Min}_i \left(\frac{x_{ij}}{a_{ij}}\right) \tag{10}$$

Thus for our simple problem we may write a set of dynamic programming return functions:

$$\begin{aligned} g_1(x_{11}, x_{21}) &= p_1 \operatorname{Min}\left(\frac{x_{11}}{a_{11}}, \frac{x_{12}}{a_{12}}\right) \\ g_2(x_{12}, x_{22}) &= p_2 \operatorname{Min}\left(\frac{x_{12}}{a_{12}}, \frac{x_{22}}{a_{22}}\right) \end{aligned} \tag{11}$$

and compute directly with (7). In this case the factor supplies b_1 and b_2 are given and are incorporated directly into (7). Results will still be supplied by the computer for factor availabilities parametric over the ranges zero to b_1 and zero to b_2 for inputs 1 and 2 respectively.

3 EQUILIBRIUM

It is not difficult to see in both of these cases that because of the optimization embodied in the recurrence relation in (7), the equilibrium conditions as specified in (2) and (4) to (6) respectively must be satisfied.

(*a*) In the optimal dynamic programming solution using the return functions in (9), the marginal value products of both factors (or of all factors in a general model) must be equal in both (all) outputs. If this were not so, there would exist a new set of values for the x's which would allow improvement in the final value of the function f, which is impossible by definition. By calculating the total factor cost, adding it back into $f_1(b_1, b_2)$, and taking first differences of the resulting function with respect to b_1 and b_2, the marginal value products of each factor in each output may be calculated under the appropriate *ceteris paribus* assumptions as to other factors and other outputs.

(*b*) In the case of the linear model, the same reasoning may be applied. A solution is obtained by dynamic programming which maximizes $(p_1y_1 + p_2y_2)$ by the definitions of equations (7), (10) and (11). This solution also satisfies (4), (and incidentally, the non-negativity condition) by virtue of the constraint on equation (7). That is, the successive application of the recurrence relation gives:

$$\begin{aligned} 0 \leqslant x_{11} + x_{12} \leqslant b_1 \\ 0 \leqslant x_{21} + x_{22} \leqslant b_2 \end{aligned} \tag{12}$$

whence, converting to an equality because we are not worrying about slack variables, we obtain at the optimum:

$$\begin{aligned} x_{11} + x_{12} = b_1 = a_{11}y_1 + a_{12}y_2 \\ x_{21} + x_{22} = b_2 = a_{21}y_1 + a_{22}y_2 \end{aligned} \tag{13}$$

which is equation (4). The satisfaction of the dual constraints in (5) follows from usual linear programming theory. Alternatively the dual problem may be solved directly by dynamic programming using an appropriate reformulation of (7). Or indeed the dual solution may be determined directly from the optimal dynamic programming solution to the primal allocation problem by taking first differences of f_1 in (7) with respect to b_1 and b_2. This will yield the required shadow prices for the two factors.

It should be pointed out that additional information may be obtained easily from these dynamic programming results. For example, the loci of combinations of b_1 and b_2 yielding given quantities of total profit may be plotted—these are isoquants of total value of output and are derivable most easily by graphical means, i.e., by drawing a graph of $f_1(b_1)$ for various levels of b_2, and then reading off combinations (b_1, b_2) yielding the required net revenue.

What do these analyses indicate? It should be apparent from them that dynamic programming applied to these models provides merely an alternative computational device for determining the required optima. It adds nothing to theory. But it is quite possible that for some specific problems of the type studied, dynamic programming might prove computationally simpler than standard methods. Alternatively one might regard the extra information which comes without cost from a dynamic programming calculation (for example results parametric over all resources) a sufficient reason for using it.[9]

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[9] In these days of vast linear programming matrices, the relevance of these statements may seem remote. It is true that for large problems there is unlikely to be a choice: "conventional" linear programming or one of its extensions would be the only possibility. It is also true that in the empirical hierarchy such problems are in a substantial majority. But research by the present author, not yet published, suggests that there are a number of significant practical constructions of the allocation problem in agriculture which can be reduced to the sorts of terms where a choice between solution techniques *is* possible. This may be achieved, still within the basic assumptions of the linear model, by more complicated formulations of dynamic programming return functions, by the addition of side conditions on recurrence relations, etc.

Dynamic programming's principal claim on our attention, however, is that unlike the two models considered above it places no restrictions on the form of the return functions g and hence, at one remove, no restrictions on the production functions w either. This is a direct result of the manner in which return functions are specified and extrema are determined in a dynamic programming calculation. In most cases functions are tabulated for regular intervals of the independent variables and maxima are located by simple search procedures.

It follows then that dynamic programming could make a theoretical contribution by allowing the derivation of equilibrium conditions for models containing basic production functions for which the requirements of standard methods are not met. Such theoretical exploration might proceed regardless of the methodological problems limiting large-scale empirical applications, since results of interest can be envisaged relating to two- or three-product or two- or three-factor firms, and in many cases generalizations to multiproduct and multifactor situations will be possible.

4 CONCLUSION

An attempt has been made in this paper to demonstrate some relationships between dynamic programming and more conventional methods of analysis in the theory of the firm. We have shown in some detail what the reader may well have regarded as self-evident from the outset, viz. that dynamic programming applied to standard production models yields exactly the same outcomes as do the traditional methods of solution. We have suggested, however, that in some examples of these problems, dynamic programming might be a preferred analytical tool. We have proceeded to point out that the production relationships specified in some models may be such that dynamic programming provides the only workable algorithm. In such circumstances, although empirical solution may be severely limited by methodological roadblocks, the possibilities for theoretical investigation of the structure of particular models may be considerable.