Australian Journal of Agricultural Economics, Vol. 29, No. 2 (August, 1985), pp. 142-148

A GENERALISED MAXIMIN APPROACH TO IMPRECISE OBJECTIVE FUNCTION COEFFICIENTS IN LINEAR PROGRAMS

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Modelling uncertainty, and attitudes to uncertainty, within a linear programming framework has long interested agricultural economists. An early, simple and enduring suggestion for accommodating uncertainty about the objective function coefficients (and subsequently, others) of a linear program involved defining a discrete set of possible coefficient vectors. This structure for uncertainty has been combined with various decision criteria or models of risk attitudes: maximin, MOTAD, focus loss and expected utility maximisation. In this paper, the more general case of linear partial information on the objective function coefficients is considered. That is, the objective function coefficients are no longer restricted to a set of discrete possibilities, but are assumed to lie within a convex space defined by a set of linear inequalities. A generalised maximin problem that can be solved as a linear program is formulated.

The traditional maximin programming model (see for example, McInerney 1969) with which farm management economists will be familiar has the form:[1]

$$(1) \qquad \max_{\mathbf{X},y} \quad y$$

$$\begin{aligned} \text{subject to} \quad A\mathbf{X} &\leq \mathbf{b} \\ \mathbf{Q}_k'\mathbf{X} - y &\geq \mathbf{0} \qquad k=1,\ldots k^* \\ \mathbf{X} &\geq \mathbf{0} \\ y &\text{ unconstrained.} \end{aligned}$$

Each $\mathbf{Q}_k$ defines a possible set of gross margins for the different activities; and the solution maximises the smallest $\mathbf{Q}_k'\mathbf{X}$ value, or total gross margin, subject to $A\mathbf{X} \leq \mathbf{b}$.

As a first step towards the generalisation to be presented, it is useful to extend the interpretation of this formulation. Since any convex combination of satisfied constraints (of the same direction) in a solution of a linear program is also satisfied by the solution, it is clear that the solution to formulation (1) also maximises the minimum $\mathbf{Q}'\mathbf{X}$ value when $\mathbf{Q}$ is constrained to lie in the convex space defined by the vertices, $\mathbf{Q}_k, k=1, \ldots k^*$. Thus, in effect, in solving a discrete possibilities maximin problem, a particular continuous possibilities problem is also solved. In the following section, more general continuous coefficient space problems are formulated directly and solved by linear programming.

[1] Vectors are denoted by bold characters. Upper case italicised symbols denote other matrices, and lower case italicised symbols denote scalars.

The Continuous Possibilities Case

Consider the standard linear programming problem:

$$\max_{\mathbf{X}} \quad \mathbf{C}'\mathbf{X} \tag{2}$$

$$\text{subject to} \quad A\mathbf{X} \leq \mathbf{b}, \quad \mathbf{X} \geq \mathbf{0}$$

where **X** and **C** are $(n \times 1)$ column vectors, **b** an $(m \times 1)$ column vector and A an $(m \times n)$ matrix. Suppose the objective function coefficients are not precisely known, but specified by the following linear partial information:

$$G\mathbf{C} \geq \mathbf{d} \tag{3}$$

where G is a $(k \times n)$ matrix and **d** a $(k \times 1)$ column vector. With **C** not uniquely specified, (2) cannot be solved. But the maximin decision rule suggests itself as a possible means of resolution. Despite the lack of a firm theoretical foundation for the rule, and its well-known inability to mimic certain decisions, it commands attention because of its simplicity in interpretation and in application. These characteristics of convenience carry over when the rule is applied in the present context.

The following problem is posed for solution:

$$\max_{\mathbf{X}} \quad \min_{\mathbf{C}|\mathbf{X}} \quad \mathbf{C}'\mathbf{X} \tag{4}$$

$$\text{subject to} \quad A\mathbf{X} \leq \mathbf{b}, \quad G\mathbf{C} \geq \mathbf{d}, \quad \mathbf{X} \geq \mathbf{0}$$

A 'Dualised' Linear Programming Solution Method

The maximin problem (4) is easily solved as a linear program after making use of duality theory. First note that the solution to (4) is necessarily at a vertex of the **C** space since **C** is selected, conditional on **X**, in an inner linear programming problem:

$$\min_{\mathbf{C}|\mathbf{X}} \quad \mathbf{C}'\mathbf{X} \tag{5}$$

$$\text{subject to} \quad G\mathbf{C} \geq \mathbf{d}$$

where **X** is given. On the other hand, the solution will not usually be at a vertex of the **X** space because the outer problem for **X** is not a linear program.

By replacing the inner linear program with its dual, namely:

$$\max_{\mathbf{v}|\mathbf{X}} \quad \mathbf{v}'\mathbf{d} \tag{6}$$

$$\text{subject to} \quad G'\mathbf{v} = \mathbf{X}, \quad \mathbf{v} \geq \mathbf{0}$$

the following problem is obtained:

$$\text{(7)} \qquad \max_{\mathbf{X}} \quad \max_{\mathbf{v}|\mathbf{X}} \quad \mathbf{v}'\mathbf{d}$$

$$\begin{aligned} \text{subject to} \quad A\mathbf{X} &\leq \mathbf{b} \\ G'\mathbf{v} - \mathbf{X} &= \mathbf{0} \\ \mathbf{X}, \mathbf{v} &\geq \mathbf{0} \end{aligned}$$

Since problem (7), in which $\mathbf{v}$ is chosen so that $\mathbf{v}'\mathbf{d}$ is a maximum for given $\mathbf{X}$, and $\mathbf{X}$ is chosen so that this conditional maximum is an overall maximum, is linear in $\mathbf{v}$ and $\mathbf{X}$, a simultaneous maximisation over $\mathbf{X}$ and $\mathbf{v}$ must yield the maximum value of $\mathbf{v}'\mathbf{d}$ achievable. But this is also the highest minimum value of $\mathbf{C}'\mathbf{X}$ as required in (4). Thus (7) provides a simple linear programming formulation for (4).

In solving (7), most interest will lie in the solution vector $\mathbf{X}$ and the objective value itself. The vector $\mathbf{v}$, interpreted as the marginal improvements in the worst scenario for $\mathbf{C}'\mathbf{X}$ (conditional on the optimal $\mathbf{X}$) obtainable through marginal changes in the information about $\mathbf{C}$, will generally be of lesser significance. This worst $\mathbf{C}$ scenario will be revealed in the shadow prices of the $G'\mathbf{v} - \mathbf{X} = \mathbf{0}$ constraints.

Variations and Particular Cases

Non-negativity of objective coefficients

In some cases, the linear partial information on $\mathbf{C}$ may include the non-negativity condition, $\mathbf{C} \geq \mathbf{0}$. The dual in (6) and the linear programming formulation (7) then involve inequality constraints, $G'\mathbf{v} \leq \mathbf{X}$, rather than equality constraints.

Equality information constraints

The linear partial information may itself involve equality constraints, $G\mathbf{C} = \mathbf{d}$. These can be accommodated by treating them as two inequalities, or, alternatively, by treating those $\mathbf{v}$ variables in (6) and (7) corresponding to the equality constraints as unconstrained in sign.

Infeasibility and unboundedness

With respect to $\mathbf{X}$, if there is no feasible solution for (4), then (7) has no feasible solution either. But if there is a feasible $\mathbf{X}$, (4) may have a finite solution, an unbounded solution, or no solution depending on the inner linear program.

If the inner linear program (5) is unbounded for some $\mathbf{X}$, its dual (and (7)) are necessarily infeasible (see, for example, Hadley 1962) for that $\mathbf{X}$. If the inner linear program is unbounded for all $\mathbf{X}$, (7) has no feasible solution. But if there exists any $\mathbf{X}$ for which it is bounded, then (7) has the corresponding feasible, finite solution.

On the other hand, if (5) is infeasible for any $\mathbf{X}$, it is infeasible for all $\mathbf{X}$ since $\mathbf{X}$ does not affect the constraints of (5). The solution to (4) is also infeasible. The dual of (5) for each $\mathbf{X}$ may be either infeasible or unbounded. If the latter occurs for some $\mathbf{X}$, the solution to (7) is unbounded; otherwise there is no feasible solution.

In summary, when (4) has a finite solution, (7) yields a finite solution. When there is no feasible $\mathbf{X}$ solution to (4), (7) is infeasible. When there is no feasible solution to (4) because of inconsistency in the information about the coefficients, (7) yields either an infeasible solution or an unbounded (negatively) solution. When the solution to (4) is unbounded (negatively), (7) yields an infeasible solution.

Full information

Suppose that full information is available about $\mathbf{C}$. That is, $\mathbf{C}=\mathbf{C}^*$. The linear program (7) becomes:

$$\max_{\mathbf{X},\mathbf{v}} \quad \mathbf{C}^{*\prime}\mathbf{v} \tag{8}$$

$$\begin{aligned} \text{subject to} \quad A\mathbf{X} &\leq \mathbf{b} \\ \mathbf{v}-\mathbf{X} &= \mathbf{0} \\ \mathbf{X} &\geq \mathbf{0} \\ \mathbf{v} &\text{ unconstrained in sign.} \end{aligned}$$

Here the simple constraints relating $\mathbf{v}$ and $\mathbf{X}$ allow one of the vectors to be eliminated. Eliminating $\mathbf{v}$, the problem reduces to the standard linear programming model (2) with known objective function coefficients.

Combining full and partial information

Suppose full information exists for the $(r\times 1)$ sub-vector $\mathbf{C}_1$, but only partial information for the complementary vector $\mathbf{C}_2$. That is,

$$\begin{aligned} \mathbf{C}_1 &= \mathbf{C}_1^* \\ G_2\mathbf{C}_2 &\geq \mathbf{d}_2 \end{aligned} \tag{9}$$

where there are p constraints defining $\mathbf{C}_2$. Formulating (7), partitioning $\mathbf{v}$ into $\mathbf{v}_1$ and $\mathbf{v}_2$, and again eliminating those $\mathbf{v}$ elements ($\mathbf{v}_1$) corresponding to $\mathbf{C}_1$, the linear program reduces to:

$$\max_{\mathbf{X},\mathbf{v}_2} \quad \mathbf{C}_1'\mathbf{X}_1+\mathbf{v}_2'\mathbf{d}_2 \tag{10}$$

$$\begin{aligned} \text{subject to} \quad A\mathbf{X} &\leq \mathbf{b} \\ G_2'\mathbf{v}_2-\mathbf{X}_2 &= \mathbf{0} \\ \mathbf{X},\ \mathbf{v}_2 &\geq \mathbf{0} \end{aligned}$$

In general, a linear program (2) in which r objective coefficients are fully known, and $n-r$ are known only to lie in a space defined by p constraints, can be resolved using the maximin criterion by augmenting the original problem with p extra activities and $n-r$ extra constraints.

Relationship to standard discrete space maximin models

The model presented here differs conceptually from the traditional maximin model in that discrete possibilities for the objective function coefficients are replaced by a continuous space. But in that the discrete formulation has a continuous space interpretation as indicated at the beginning of the paper, the present model also generalises the traditional formulation. The latter is the 'dualised' linear program corresponding to a particular case of the general maximin model (4), namely:

$$(11) \qquad \max_{\mathbf{X}} \quad \min_{\mathbf{C}|\mathbf{X}} \quad \mathbf{C}_1'\mathbf{X}_1 + \mathbf{C}_2'\mathbf{X}_2$$

$$\text{subject to} \quad A\mathbf{X} \leq \mathbf{b}$$

$$\begin{bmatrix} I & -V \\ \mathbf{0}' & \mathbf{i}' \end{bmatrix} \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}$$

$$\begin{aligned} \mathbf{X} &\geq \mathbf{0} \\ \mathbf{C}_2 &\geq \mathbf{0} \\ \mathbf{X}_2 &= \mathbf{0} \\ \mathbf{C}_1 &\ \text{unconstrained.} \end{aligned}$$

where $\mathbf{i}$ is a unit vector. The $\mathbf{X}_2$ vector of activities could be dropped from (11) because it is identically null. Its inclusion serves only to emphasise that (11) is a particular case of (4).

The dual of the inner problem is:

$$(12) \qquad \max_{\mathbf{d}_V, d|\mathbf{X}} \quad \mathbf{0}'\mathbf{d}_V + d$$

$$\text{subject to} \quad \begin{bmatrix} I & \mathbf{0} \\ -V' & \mathbf{i} \end{bmatrix} \begin{bmatrix} \mathbf{d}_V \\ d \end{bmatrix} \begin{Bmatrix} = \\ \leq \end{Bmatrix} \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}$$

$$\mathbf{d}_V,\ d \ \text{unconstrained.}$$

The first set of equations establishes direct correspondence between $\mathbf{d}_V$ and $\mathbf{X}_1$. Eliminating $\mathbf{d}_V$, removing the null vector $\mathbf{X}_2$ from the model and partitioning A, yields the 'dualised' problem:

$$(13) \qquad \max_{\mathbf{X}_1, d} \quad d$$

$$\text{subject to} \quad \begin{aligned} A_1\mathbf{X}_1 &\leq \mathbf{b} \\ V'\mathbf{X}_1 - \mathbf{i}d &\geq \mathbf{0} \\ \mathbf{X}_1 &\geq \mathbf{0} \\ d &\ \text{unconstrained.} \end{aligned}$$

Apart from notational differences, this is precisely the problem formulated in (1).

It is worth emphasising that the dependence between activity gross margins, captured in the traditional model by specifying possible combinations of gross margin values, is not lost in the generalised continuous space model. Dependence is captured directly by specifying linear relations between gross margins.

Although the traditional model (1) can be viewed as the 'dualised' formulation of one case of the general maximin model (4), it is also true that any problem (4) can be solved as a problem of type (1). In particular, the $\mathbf{Q}_k$'s of (1) have to be selected as the vertices of the feasible space for $\mathbf{C}$, namely the vertices of $G\mathbf{C} \geq \mathbf{d}$. But this would represent an inefficient means of solving (4) because of the large number of vertices which would have to be explicitly located and included as constraints.

Relationship to game theory

The problem addressed here can be regarded as a two-player game. But it differs from the standard game in two respects. First, the actions available to each player are not limited to a finite discrete set, but form the continuous, linearly defined **X** and **C** spaces. Second, there is asymmetrical information in that one player (**C** selector) is assumed to know the other's selection. Thus the former's optimal strategy is the pure one of selecting the action (**C**) known to be best for himself. Similarly, knowing this strategy, the second player's optimal strategy will also be a pure strategy, some particular **X** value.

These pure strategies contrast with the mixed (or probabilistic) strategies of the common solution of the standard matrix game (for example, von Neumann and Morgenstern 1947; Hadley 1962).

Nevertheless, there is some algebraic similarity between the problems. Any choice of **X** can be regarded as a selection of a convex combination (but no longer probabilistic) of the discrete actions represented by the vertices of the **X** space. The same holds for any **C** choice. A matrix of payoffs can be constructed from the **X** vertex-**C** vertex combinations, and the players' problem defined as that of selecting normalised weights to place on their vertices. The **X** player is free to select any combination, but the **C** player is confined to collapsed weights such that all weight must be on one vertex. Algebraically, the present model has the form of a restricted version of the standard matrix game.

Concluding Remarks

The generalised maximin formulation has been introduced in the context of farm planning under gross margin uncertainty. But before it can be applied in a realistic farm planning situation, research is needed to determine how well uncertainty can be represented through linear partial information and to determine how best to elicit such information. A tutorial example of the maximin formulation is presented in Drynan (1984).

However, the model is not confined to farm planning problems. It applies to any linear programming model for which there is only partial linear information about the objective function coefficients. Its applicability is limited only by access to this kind of information and by the appropriateness of a maximin criterion.

The model is more immediately applicable in contexts where the linear partial information is objective. Drynan (1983) has shown how the detection of dominated cash flow streams under various informational assumptions about the discount factors reduces to a maximin problem. Drynan and Sandiford (1985) have applied the model in a 'minimax deviation' goal programming analysis of the Scottish inshore fishery. Optimal fishery policies were derived under different assumptions concerning the level of information about the trade-offs between cost, regional employment, and catch objectives.

Even within the field of planning under uncertainty, the potential of the model extends beyond maximising minimum income. Drynan

(1984) outlines how to derive a bound on the cumulative distribution function for total gross margin in a wait-and-see stochastic programming model, how to determine the worst gross margin scenario that could confront the decision maker and how to handle some forms of uncertainty in constraints.

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