Does Price Cause Demand or Vice Versa?

Evidence from Demand Analyses for Soft Drinks in the US

Dae-Heum Kwon
Research Assistant Professor
Department of Agribusiness and Applied Economics
North Dakota State University
Email: heum_us@yahoo.com
Phone: (701) 231-8993
Fax: (701) 231-7400

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I. Introduction

Since the mixed demand system (Samuelson, 1965) was introduced, full spectrums of model specifications exist for demand analyses. While the direct (inverse) demand system specifies quantity demanded (willingness to pay) as a function of prices (quantities), the mixed demand system specifies demand relationships as a function of mixed set of prices and quantities. However, the specification choice is usually based on researchers' intuition about the product properties or market characteristics of a specific commodity and the empirical comparisons are rarely pursued. As Thurman (1986) argued "it is odd that such arguments rest solely on a priori notions" and the coexistence of alternative specification can result in ambiguities. For example, both the direct (Wohlgenant and Hahn, 1982) and the inverse (Shonkwiler and Taylor, 1984) demand functions are used for poultry market data. Stockton, Capps, and Bessler (SCB) (2005) propose the Causally-Identified Demand System (CIDS) to address this issue by using the empirically inferred local causal structures based on the graphical causal model (Pearl, 2000). More specifically, SCB use the PC algorithm to infer local causal structures among price and quantity variables for meat consumption and demonstrate that the Rotterdam mixed demand system identified through the application of the PC algorithm is statistically preferred to the synthetic direct demand systems.

The objective of this paper is to extend their approach in several ways for the full use of the direct, inverse, and mixed demand systems (three alternative demand specifications). First, the synthetic functional form is derived for the mixed demand system in order to minimize the effect of functional form for comparisons among three alternative specifications. SCB use the Rotterdam functional form for the mixed demand system derived by Moschini and Vissa (1993). However, the Rotterdam type parameterization a priori assumes that the marginal expenditure shares and Slutsky terms are constant. The derived synthetic mixed demand system nests the

Rotterdam, LA/AIDS, NBR, CBS forms (four differential functional forms) and extends the functional form of Matsuda (2004), which nests Rotterdam and CBS. Second, the empirical methods to compare alternative demand specifications are proposed. Alternative specifications are non-nested each other and have different dependent variables. To address these issues, (i) the model selection approaches such as the Likelihood Dominance Criterion (Pollak and Wales, 1991) are pursued and (ii) the synthetic functional forms of three specifications are further transformed to have the common differential AIDS type dependent variable. These generalized functional forms of three specifications allow the model selection comparisons among three specifications and extend the results of Eales, Durham, and Wessells (1997), which pursue a convenient comparison between the direct and inverse demand system. Third, the complete relationships among three specifications are derived to allow convenient comparisons of the elasticities/flexibilities estimated from alternative specifications. For example, the elasticities form in the direct specification can be retrieved from the estimates of the inverse and mixed demand system. The derived relationships extend the identified relationships between direct and mixed demand systems of Moschini and Vissa (1993) to those between inverse and mixed demand systems as well as direct and inverse demand systems. Finally, to more fully incorporate the graphical causal model for demand analysis, the Greedy Equivalence Search (GES) algorithm is additionally introduced and compared with the PC algorithm used in SCB. Note that the graphical causal model has been developed in two distinctive approaches of conditional independent test approach (PC algorithm) and goodness-of-fit scoring approach (GES algorithm). The argued advantages of GES algorithm relative to PC algorithm are empirically tested.

Based on the extensions mentioned above, we propose the following procedure: (i) the causal structures are inductively inferred based on the graphical causal models of the PC and

GES algorithms. The information of local causal structure provides guidance for the specification choice among the direct, inverse, and mixed demand functions; (ii) the inferred specifications are estimated with the generalized functional forms, which extend the synthetic approach based on the differential functional form framework; and (iii) the comparison of alternative specifications is conducted in terms of the common elasticities/flexibilities estimated/retrieved from alternative specifications and model selection approach. The proposed method is applied for soft drink consumption by using retail checkout scanner data from Dominick' Finer Foods.

II. Empirical Procedures

Graphical Causal Models: PC and GES algorithms

Given that theory does not provide enough information for the choice among direct, inverse, and mixed demand systems, the specification choice is usually based on researchers' intuition about product properties or market characteristics of a specific commodity. The typical arguments for quantity-dependent specification rely on the price-taking agent assumption, the short-run fixity in prices, or the administratively setting of price in publicly offered goods. On the other hand, the usual arguments for price-dependent specification are based on the biological lags in production and the non-storable or perishable properties of commodities, or the Bertrand type strategic pricing rules of suppliers in differentiated good. More fundamentally, the specification choice is closely related with the identification issue of the local causal structure between price and quantity for a specific commodity. When we choose quantity-dependent (price-dependent) specification, we implicitly assume a local causal structure that the price (quantity) causes the quantity (price) variable. From this perspective, the graphical causal models provide an alternative empirical method for the choice among the direct, inverse, and mixed demand systems (Stockton, Capps, and Bessler, 2005). The first step in empirical modeling the consumer behavior is applying the graphical causal models for the price and quantity variables for the relevant commodities as well as their total expenditure variables. The local information of the identified causal structure provides the empirical guidance for the specification choice among direct, inverse, and mixed demand functions.

Although the graphical causal method is introduced in some econometric literatures (e.g., Swanson and Granger 1997, Bessler and Yang 2003, Hoover 2005), its potential advantages are not fully recognized. Furthermore, the previous applications of the graphical causal model often rely on the PC algorithm, while this study uses the more recent approach of GES algorithm as well. On this reason, we provide a brief explanation of the graphical causal models on the next section. We refer to Spirtes et al. (2000) for the detailed information of the PC algorithm, developed from the conditional independence test approach. The GES algorithm, developed from the goodness-of-fit (Bayesian) scoring perspective, is originated from Meek (1997) and its optimality is proved by Chickering (2003). More theoretical and conceptual aspects of graphical causal models are explained by Pearl (2000).

The graphical causal models have been developed by mathematically connecting probabilistic structures to graphical concepts, which effectively and efficiently capture all the probabilistic structures in data. The graphical causal model or directed acyclic graph (DAG) approaches are based on several mathematical propositions. When it is assumed that the cyclic or feedback causal structure does not exist (causal acyclic condition) and all the causally relevant variables can be measured (causal sufficiency condition), it is proved that the probability distribution follows the Markov condition such that every variable is independent of all its causal nondescendants, conditional on its direct cause (Pearl and Verma, 1991). This implies that (i) an effect is independent of its indirect causes conditional on its direct causes, and (ii) the effect variables are independent conditional on their common causes. For example, two variables A and B in both the causal chain ($A \rightarrow C \rightarrow B$ or $A \leftarrow C \leftarrow B$) and fork ($A \leftarrow C \rightarrow B$) are unconditionally dependent on each other, but conditionally independent given C . On the other hand, the other logically possible causal structure among three connected variables is known as the selection bias (unshielded-collider of $A \rightarrow C \leftarrow B$), where observation on a common consequence of two unconditionally independent causes tends to make those two causes dependent conditional on common effect (Kim and Pearl, 1983). This causal structure of the unshielded-collider provides an "empirical clue" to address induction problem that correlation does not imply causation. The combinational statistical information of marginal correlation (unconditionally independence of A and B) and partial correlation (conditional dependence of A and B given C) makes it possible to infer the causal structure of the unshielded-collider, which is discriminated from the observational equivalent causal structures of the causal chain and fork. Based on the empirical clue of the unshielded-collider, the graph theory in the graphical causal model plays two important roles to infer the underlying causal structures. (i) The graph theory provides mathematical information to logically decide relevant search spaces and allows an efficient use of the maximum information of (un)conditional probabilistic structures from the data. Without such systematically and efficiently defining the relevant or entire search space, checking or searching all the relevant (un)conditional probabilistic structures among all the possible combinations of variables becomes infeasible. (ii) The graph theory also provides logical orientation rules to partially discriminate the observationally equivalent causal structures. The logical inferences about causal directions are based on the following idea: An orienting the remaining undirected edges does not result in the causal structure which is inconsistent with the statistical observations, as long as the logically decided orientations do not create either the new unshielded-collider structure or the cyclic causal structure. The former structure is empirically unsupported by data and the latter one is logically excluded by the acyclic assumption (Verma and Pearl 1992, Meek 1995b).

To empirically infer the (un)conditional probabilistic structures, two distinctive approaches have been proposed: conditional independent test and goodness-fit scoring approaches. The conditional independence test approach, incorporated in the PC algorithm, is based on the qualitative decisions about local independence tests. However, it is not easy to decide the appropriate significance level for the local tests, because the power of algorithm against alternatives is an extremely complex and unknown function of the power of the individual local tests. Thus, the PC algorithm can be susceptible to incorrect qualitative local decisions and may provide the sensitive results to the chosen significant level. On the other hand, the goodness-of-fit scoring approach, incorporated in the GES algorithm, does not require choosing a specific significance level and may provide finer results. It is because the goodness-of-fit scoring approach is based on the quantitative measure about how much the overall independence constraints associated with an entire causal structure are true. The GES algorithm uses the Bayesian Information Criterion (BIC) as a measure of scoring goodness-fit of a given DAG G at each search step. The BIC is chosen as a goodness-fit score because (i) it is a consistent approximation of the Bayesian posterior probability under the Gaussian and multinomial distributions and (ii) it has *decomposability* and *equivalence* properties, that allow efficient scoring. BIC for a given DAG G of a set of variables $V = \{X_1, \cdots, X_N\}$ can be written as follows: $BIC(V,G) = \log P(V \mid G) - \dim(G) \cdot \log(T)/2 \cdot$, where T is the sample size, $\dim(G)$ is the dimension or the number of parameters of DAG G, and $\log P(V \mid G)$ is the log-likelihood function for a set of variables V given DAG G. For a given DAG G at each step of the search procedures, the $\log P(V \mid G)$ can be efficiently evaluated by using *decomposable* property of

$\log P(X_1, \ldots, X_N) = \sum_n \log P(X_n | Pa_n)$, where Pa_n represent direct causal parents. The *equivalent* property of BIC scores comes from the fact that DAGs in an equivalence class have the same number of edges and a common factorization. For example, the joint distribution of $P(A, B, C)$ can be factorized as $P(C | A)P(A | B)P(B)$ and $P(B | A)P(C | A)P(A)$ for DAG of $C \leftarrow A \leftarrow B$ and $C \leftarrow A \rightarrow B$, respectively. The relationship $P(A | B)P(B) = P(A, B) = P(B | A)P(A)$ by the Bayesian theorem makes the two DAGs equivalent. It is demonstrated that under the Gaussian and multinomial distributions, this independence equivalence become identical to distributional equivalence, which means that equivalence class of DAGs have the same probability distribution.

Functional Forms of Direct, Inverse, and Mixed Demand Systems

After obtaining the empirical guidance for the specification choice among the direct, inverse, and mixed demand functions through the application of the graphical causal models, the next step is to estimate the functional relationships among the price and quantity variables for the relevant commodities as well as their total expenditure variables. Various functional forms are used for the direct and inverse demand systems. However, when we want to compare the direct, inverse, and mixed demand systems with minimizing the effect of functional specifications, the possible use of the mixed demand system imposes some limitations for considering possible range of functional forms. It is because the mixed demand system requires consistent and simultaneous specifications for both direct and indirect utility functions and the commonly used flexible functional forms, such as the Translog and the Almost Ideal Demand Systems (AIDS), do not have a closed form dual representation for both direct and indirect utility functions. As Moschini and Vissa (1993) emphasize, an appropriate approach for a flexible demand system of mixed demand functions is to approximate each demand function directly by a differential Rotterdam demand system and to impose the theoretical restrictions.

However, the parameterization assumptions for the Rotterdam functional form has different implications for the empirical results, comparing with those for another commonly used functional form of AIDS (Lee, Brown, and Seal, 1994). While the Rotterdam functional form assumes that both the expenditure (scale) coefficient and the compensated price (quantity) coefficient in the direct (inverse) demand system are constant parameters, the AIDS or Linear Approximated AIDS (LA/AIDS) functional form assumes that both are function of budget shares. Two more logically possible combinations of constant/variational parameterization for these two coefficients are also used for both the direct and inverse systems. While Keller and van Driel (1985) of Dutch Central Bureau of Statistics (CBS) introduce the variational expenditure (scale) coefficient with the constant Slutsky (Antonelli) coefficient by reparameterizing the Rotterdam specification, Neves (1987) of Netherlands National Bureau of Research (NBR) introduce the constant income (scale) coefficient with the variational Slutsky (Antonelli) coefficient by reparameterizing the LA/AIDS specification.

To address the issue that the elasticities (flexibilities) are sensitive to the chosen parameterizations among the Rotterdam, LA/AIDS, and two hybrid demand specifications of CBS and NBR in the direct and inverse demand systems, Barten (1993) and Brown, Lee, and Seal (1995) propose the synthetic functional form for the direct and inverse demand system respectively, based on the principle of artificial nesting. The synthetic functional form nests the four differential families and the statistical tests of the nesting parameters provide the empirical guidance for the best parameterization among the differential family of functional forms. Furthermore, it has been demonstrated that these two synthetic direct and inverse demand systems can be considered as demand systems in their own right, beyond an artificial composite of known models. For example, Matsuda (2005) shows that one of the nesting coefficients in the inverse synthetic model of Brown, Lee, and Seal (1995) implies the transformation parameter of the Box-Cox scale curves.

Applying the similar approach for the mixed demand systems, Matsuda (2004) extends the Rotterdam parameterizations of Moschini and Vissa (1993) by incorporating a generalized form of marginal budget shares. However, comparing with the synthetic functional form for the direct and inverse demand system, Matsuda's functional form only encompasses the Rotterdam and CBS specifications, not the LA/AIDS and NBR specifications. To fill out this gap, the synthetic differential demand model is derived for the mixed demand system based on the similar logic to derive synthetic demand model in direct and inverse demand systems. The derived synthetic mixed demand system allows estimating the direct, inverse, and mixed demand systems in the similar degrees of flexibility in functional form specifications, when the flexibility means the capability of the empirical model to allow the possible combinations of constant/variational parameterization for the expenditure (scale) and the Slutsky (Antonelli) coefficients.

Within the direct, inverse, or mixed demand system, the statistical tests of the two nesting coefficients provide the empirical guidance for the best parameterizations among the Rotterdam, LA/AIDS, CBS, NBR, and synthetic functional forms. On the other hand, the empirical comparisons across three different specifications are still difficult, because the direct, inverse, and mixed demand systems have different dependent variables. To address these issues, based on the approach of Eales, Durham, and Wessells (1997), the synthetic functional forms of three specifications are further transformed to have the common differential AIDS type dependent variable. These generalized functional forms of three specifications allow easy comparisons among three specifications and extend the results of Eales, Durham, and Wessells (1997), which pursue the comparison between the direct and inverse demand system.

The main advantage of these differential functional form approaches is that the four differential functional forms, synthetic, and generalized functional forms can be directly derived from the Rotterdam demand system, which is regarded as flexible in that it provides a first-order approximation to an arbitrary demand system in either parameter or variable space (Mountain, 1988). Thus this approach does not need to specify the direct and/or indirect utility functions. Let the set of commodities of interest $A \cup B = \{1, \cdots, m, m+1, \cdots, N\}$ be divided into quantity-dependent $A = \{1, \cdots, m\}$ and price-dependent $B = \{m+1, \cdots, N\}$ commodity groups. The subscripts $(n, n') \in A \cup B$, $(i, j) \in A$, and $(k, r, s) \in B$ are used to denote whole and each group of commodities, respectively. Total expenditure and the normalized prices can be represented by $y \equiv P \cdot Q \equiv P_A \cdot Q_A + P_B \cdot Q_B$ and $\pi_n = p_n / y$, respectively. The superscript c is used for compensation and D, I, and M are used for the direct, inverse, and mixed demand systems, respectively. The $\delta_{n,n'}$ denotes the Kronecker delta such that $\delta_{n,n'} = 1$ for $n = n'$ and $\delta_{n,n'} = 0$ for $n \neq n'$. Both the relationships among the Rotterdam, LA/AIDS, CBS, NBR functional forms and their connections to the synthetic and generalized functional forms are based on the differential relationships of $dw_n = \{w_n d \ln q_n\} + w_n d \ln p_n - w_n d \ln y$, $dw_n = \{w_n d \ln \pi_n\} + w_n d \ln q_n$, and $d \ln y = d \ln \bar{y} + d \ln p_A$ for the direct, inverse, and mixed demand specification, respectively. Derivations of the synthetic and generalized functional forms for the direct, inverse, mixed demand functions are explained in Appendix A. The synthetic and generalized functional forms can be summarized as follows. The original form of four differential forms and their corresponding Rotterdam- and AIDS-type dependent variables forms are provided to clarify their relationships with the synthetic and generalized functional forms.

The differential family of four direct demand systems can be summarized and nested in the synthetic or generalized direct demand systems. If the expenditure coefficient is defined as $a_n \equiv [w_n \varepsilon_n]$ or $c_n \equiv [w_n \varepsilon_n - w_n]$ and the Slutsky coefficient is defined as $a_{n,n'} \equiv [w_n \varepsilon^c_{n,n'}]$ or $c_{n,n'} \equiv [w_n \varepsilon^c_{n,n'} - w_n (w_{n'} - \delta_{n,n'})]$, then both are nested by the synthetic parameters of $C_n \equiv [w_n \varepsilon_n - \theta_1^D w_n]$ and $C_{n,n'} \equiv [w_n \varepsilon^c_{n,n'} - \theta_2^D w_n (w_{n'} - \delta_{n,n'})]$, respectively.

Rotterdam :

$$w_n d \ln q_n = [w_n \varepsilon_n] d \ln Q + \sum_{n'=1}^{N} [w_n \varepsilon^c_{n,n'}] d \ln p_{n'} \text{ or}$$

$$w_n d \ln q_n = [a_n] d \ln Q + \sum_{n'=1}^{N} [a_{n,n'}] d \ln p_{n'} \text{ or}$$

$$dw_n = [a_n - w_n] d \ln Q + \sum_{n'=1}^{N} [a_{n,n'} - w_n (w_{n'} - \delta_{n,n'})] d \ln p_{n'} .$$

Differential AIDS:

$$dw_n = [w_n \varepsilon_n - w_n] d \ln Q + \sum_{n'=1}^{N} [w_n \varepsilon^c_{n,n'} - w_n (w_{n'} - \delta_{n,n'})] d \ln p_{n'} \text{ or}$$

$$w_n d \ln q_n = [c_n + w_n] d \ln Q + \sum_{n'=1}^{N} [c_{n,n'} + w_n (w_{n'} - \delta_{n,n'})] d \ln p_{n'} \text{ or}$$

$$dw_n = [c_n] d \ln Q + \sum_{n'=1}^{N} [c_{n,n'}] d \ln p_{n'} .$$

CBS:

$$w_n d \ln\left(\frac{q_n}{Q}\right) = [c_n] d \ln Q + \sum_{n'=1}^{N} a_{n,n'} \ln p_{n'} \text{ or}$$

$$w_n d \ln q_n = [c_n + w_n] d \ln Q + \sum_{n'=1}^{N} [a_{n,n'}] d \ln p_{n'} \text{ or}$$

$$dw_n = [c_n] d \ln Q + \sum_{n'=1}^{N} [a_{n,n'} - w_n (w_{n'} - \delta_{n,n'})] d \ln p_{n'} .$$

NBR:

$$(dw_n + w_n d \ln Q) = [a_n] d \ln Q + \sum_{n'=1}^{N} [c_{n,n'}] d \ln p_{n'} \text{ or}$$

$$w_n d \ln q_n = [a_n] d \ln Q + \sum_{n'=1}^{N} [c_{n,n'} + w_n (w_{n'} - \delta_{n,n'})] d \ln p_{n'} \text{ or}$$

$$dw_n = [a_n - w_n] d \ln Q + \sum_{n'=1}^{N} [c_{n,n'}] d \ln p_{n'} .$$

Synthetic: $$w_n d\ln q_n = \left[C_n + \theta_1^D w_n\right] d\ln Q + \sum_{n'=1}^{N}\left[C_{n,n'} + \theta_2^D w_n\left(w_{n'} - \delta_{n,n'}\right)\right] d\ln p_{n'}$$

Generalized: $$dw_n = \left[C_n - \left(1-\theta_1^D\right) w_n\right] d\ln Q + \sum_{n'=1}^{N}\left[C_{n,n'} - \left(1-\theta_2^D\right) w_n\left(w_{n'} - \delta_{n,n'}\right)\right] d\ln p_{n'} \; .$$

Theoretical restrictions can be imposed by using following relations

(a) Homogeneity: $\sum_{n'=1}^{N} C_{n,n'} = 0$,

(b) Symmetry: $C_{n,n'} = C_{n',n}$,

(c) Adding-up: $\sum_{n=1}^{N} C_n = 1-\theta_1^D$.

The elasticities can be calculated as follows

(a) Expenditure elasticity: $\varepsilon_n = \dfrac{C_n}{w_n} + \theta_1^D$,

(b) Compensated elasticity: $\varepsilon_{n,n'}^c = \dfrac{C_{n,n'}}{w_n} + \theta_2^D\left(w_{n'} - \delta_{n,n'}\right)$, and

(c) Uncompensated elasticity: $\varepsilon_{n,n'} = \left[\dfrac{C_{n,n'}}{w_n} + \theta_2^D\left(w_{n'} - \delta_{n,n'}\right)\right] - \left[C_n\left(\dfrac{w_{n'}}{w_n}\right) + \theta_1^D w_{n'}\right]$.

The differential family of four inverse demand systems can be summarized and nested in the synthetic or generalized inverse demand systems. When the scale coefficient is defined as $b_n \equiv \left[w_n f_n\right]$ or $d_n \equiv \left[w_n f_n + w_n\right]$ and the Antonelli coefficient is defined as $b_{n,n'} \equiv \left[w_n f_{n,n'}^c\right]$ or $d_{n,n'} \equiv \left[w_n f_{n,n'}^c - w_n\left(w_{n'} - \delta_{n,n'}\right)\right]$, both of them are nested by the synthetic parameters of $D_n \equiv \left[w_n f_n + \theta_1^I w_n\right]$ and $D_{n,n'} \equiv \left[w_n f_{n,n'}^c - \theta_2^I w_n\left(w_{n'} - \delta_{n,n'}\right)\right]$ respectively.

Rotterdam: $w_n d\ln \pi_n = \left[w_n f_n\right] d\ln Q + \sum_{n'=1}^{N}\left[w_n f_{n,n'}^c\right] d\ln q_{n'}$ or

$w_n d\ln \pi_n = \left[b_n\right] d\ln Q + \sum_{n'=1}^{N}\left[b_{n,n'}\right] d\ln q_{n'}$ or

$$dw_n = [b_n + w_n]d\ln Q + \sum_{n'=1}^{N}[b_{n,n'} - w_n(w_{n'} - \delta_{n,n'})]d\ln q_{n'} .$$

Differential AIDS: $$dw_n = [w_n f_n + w_n]d\ln Q + \sum_{n'=1}^{N}[w_n f^c_{n,n'} - w_n(w_{n'} - \delta_{n,n'})]d\ln q_{n'} \text{ or}$$

$$w_n d\ln\pi_n = [d_n - w_n]d\ln Q + \sum_{n'=1}^{N}[d_{n,n'} + w_n(w_{n'} - \delta_{n,n'})]d\ln q_{n'} \text{ or}$$

$$dw_n = [d_n]d\ln Q + \sum_{n'=1}^{N}[d_{n,n'}]d\ln q_{n'} .$$

CBS: $$w_n d\ln\left(\frac{p_n}{P}\right) = [d_n]d\ln Q + \sum_{n'=1}^{N}[b_{n,n'}]\ln q_{n'} \text{ or}$$

$$w_n d\ln\pi_n = [d_n - w_n]d\ln Q + \sum_{n'=1}^{N}[b_{n,n'}]d\ln q_{n'} \text{ or}$$

$$dw_n = [d_n]d\ln Q + \sum_{n'=1}^{N}[d_{n,n'}]d\ln q_{n'} .$$

NBR: $$(dw_n + w_n d\ln Q) = [b_n]d\ln Q + \sum_{n'=1}^{N}[d_{n,n'}]d\ln p_{n'} \text{ or}$$

$$w_n d\ln\pi_n = [b_n]d\ln Q + \sum_{n'=1}^{N}[d_{n,n'} + w_n(w_{n'} - \delta_{n,n'})]d\ln q_{n'} \text{ or}$$

$$dw_n = [b_n + w_n]d\ln Q + \sum_{n'=1}^{N}[c_{n,n'}]d\ln p_{n'} .$$

Synthetic: $$w_n d\ln\pi_n = [D_n - \theta_1^I w_n]d\ln Q + \sum_{n'=1}^{N}[D_{n,n'} + \theta_2^I w_n(w_{n'} - \delta_{n,n'})]d\ln q_{n'}$$

Generalized: $$dw_n = [D_n + (1 - \theta_1^I)w_n]d\ln Q + \sum_{n'=1}^{N}[D_{n,n'} - (1 - \theta_2^I)w_n(w_{n'} - \delta_{n,n'})]d\ln q_{n'} .$$

Theoretical restrictions can be imposed by using following relations

(a) Homogeneity: $\sum_{n'=1}^{N} D_{n,n'} = 0$,

(b) Symmetry: $D_{n,n'} = D_{n',n}$,

(c) Adding-up: $\sum_{n=1}^{N} D_n = -1 + \theta_1^I$.

The elasticities can be calculated as follows

(a) Scale flexibility: $f_n = \frac{D_n}{w_n} - \theta_1^I$,

(b) Compensated flexibility: $f_{n,n'}^c = \frac{D_{n,n'}}{w_n} + \theta_2^I\left(w_{n'} - \delta_{n,n'}\right)$, and

(c) Uncompensated flexibility: $f_{n,n'} = \left[\frac{D_{n,n'}}{w_n} + \theta_2^I\left(w_{n'} - \delta_{n,n'}\right)\right] + \left[D_n\left(\frac{w_{n'}}{w_n}\right) - \theta_1^I w_{n'}\right]$.

The differential family of mixed demand systems can be derived and nested in either Rotterdam or AIDS dependent variable forms of analogous synthetic mixed demand systems, when the expenditure coefficients of group A and B are defined as $\alpha_i \equiv \left[w_i\varepsilon_i - \theta_1^M w_i\right]$ and $\beta_k \equiv \left[w_k f_k - \theta_1^M w_k\right]$ and the Slutsky coefficients are defined as $\alpha_{i,j} \equiv \left[w_i\varepsilon_{i,j}^c - \theta_2^M w_i\left(w_j - \delta_{i,j}\right)\right]$, $\beta_{k,s} \equiv \left[w_k f_{k,s}^c - \theta_2^M w_k\left(w_s - \delta_{k,s}\right)\right]$, $-\gamma_{k,j} \equiv \left[w_k p_{k,j}^c + \theta_2^M w_k w_j\right]$, and $g_{i,s} \equiv \left[w_i q_{i,s}^c - \theta_2^M w_i w_s\right]$.

Rotterdam:

$$
\begin{aligned}
w_i d\ln q_i = {} & \left[w_i\varepsilon_i\right]\cdot d\ln\bar{y} \\
& + \sum_{j=1}^{m}\left[w_i\varepsilon_{i,j}^c - w_i\varepsilon_i\cdot\left(\sum_{k=m+1}^{N} w_k\cdot p_{k,j}^c\right)\right]\cdot d\ln p_j \\
& + \sum_{s=m+1}^{N}\left[w_i q_{i,s}^c - w_i\varepsilon_i\cdot\left(\sum_{r=m+1}^{N} w_r f_{r,s}^c\right)\right]\cdot d\ln q_s
\end{aligned}
$$

$$
\begin{aligned}
w_k d\ln p_k = {} & \left[w_k f_k\right]\cdot d\ln\bar{y} \\
& + \sum_{j=1}^{m}\left[w_k p_{k,j}^c - w_k f_k\cdot\left(\sum_{r=m+1}^{N} w_r p_{r,j}^c\right)\right]\cdot d\ln p_j \\
& + \sum_{s=m+1}^{N}\left[w_k f_{k,s}^c - w_k f_k\cdot\left(\sum_{r=m+1}^{N} w_r f_{r,s}^c\right)\right]\cdot d\ln q_s
\end{aligned}
$$

Synthetic:

$$
\begin{aligned}
w_i d\ln q_i = {} & \left[\alpha_i + \theta_1^M w_i\right]\cdot d\ln\bar{y} \\
& + \sum_{j=1}^{m}\left[\alpha_{i,j} + \theta_2^M w_i\left(w_j - \delta_{i,j}\right) - \left(\alpha_i + \theta_1^M w_i\right)\cdot\left(\sum_{r=m+1}^{N} -\gamma_{r,j} - \theta_2^M w_r w_j\right)\right]\cdot d\ln p_j \\
& + \sum_{s=m+1}^{N}\left[g_{i,s} + \theta_2^M w_i w_s - \left(\alpha_i + \theta_1^M w_i\right)\cdot\left(\sum_{r=m+1}^{N} \beta_{r,s} + \theta_2^M w_r\left(w_s - \delta_{r,s}\right)\right)\right]\cdot d\ln q_s
\end{aligned}
$$

$$
\begin{aligned}
w_k d\ln p_k &= \left[\beta_k + \theta_1^M w_k\right]\cdot d\ln \bar{y} \\
&\quad + \sum_{j=1}^{m}\left[-\gamma_{k,j} - \theta_2^M w_k w_j - \left(\beta_k + \theta_1^M w_k\right)\cdot\left(\sum_{r=m+1}^{N} -\gamma_{r,j} - \theta_2^M w_r w_j\right)\right] d\ln p_j \\
&\quad + \sum_{s=m+1}^{N}\left[\beta_{k,s} + \theta_2^M w_k\left(w_s - \delta_{k,s}\right) - \left(\beta_k + \theta_1^M w_k\right)\cdot\left(\sum_{r=m+1}^{N}\beta_{r,s} + \theta_2^M w_r\left(w_s - \delta_{r,s}\right)\right)\right] d\ln q_s
\end{aligned}
$$

Generalized:

$$
\begin{aligned}
dw_i &= \left[\alpha_i + \left(\theta_1^M - 1\right) w_i\right]\cdot d\ln \bar{y} \\
&\quad + \sum_{j=1}^{m}\left[\alpha_{i,j} + \left(\theta_2^M - 1\right) w_i\left(w_j - \delta_{i,j}\right) - \left(\alpha_i + \theta_1^M w_i\right)\cdot\left(\sum_{r=m+1}^{N} -\gamma_{r,j} - \theta_2^M w_r w_j\right)\right]\cdot d\ln p_j \\
&\quad + \sum_{s=m+1}^{N}\left[g_{i,s} + \theta_2^M w_i w_s - \left(\alpha_i + \theta_1^M w_i\right)\cdot\left(\sum_{r=m+1}^{N}\beta_{r,s} + \theta_2^M w_r\left(w_s - \delta_{r,s}\right)\right)\right]\cdot d\ln q_s
\end{aligned}
$$

$$
\begin{aligned}
dw_k &= \left[\beta_k + \left(\theta_1^M - 1\right) w_k\right]\cdot d\ln \bar{y} \\
&\quad + \sum_{j=1}^{m}\left[-\gamma_{k,j} - \left(\theta_2^M + 1\right) w_k w_j - \left(\beta_k + \theta_1^M w_k\right)\cdot\left(\sum_{r=m+1}^{N} -\gamma_{r,j} - \theta_2^M w_r w_j\right)\right]\cdot d\ln p_j \\
&\quad + \sum_{s=m+1}^{N}\left[\beta_{k,s} + \theta_2^M w_k w_s + \left(1 - \theta_2^M\right) w_k \delta_{k,s} - \left(\beta_k + \theta_1^M w_k\right)\cdot\left(\sum_{r=m+1}^{N}\beta_{r,s} + \theta_2^M w_r\left(w_s - \delta_{r,s}\right)\right)\right]\cdot d\ln q_s
\end{aligned}
$$

Theoretical restrictions can be imposed by using following relations

(a) Homogeneity: $\sum_{j=1}^{m}\alpha_{i,j} = \theta_2^M w_i\left(1 - \sum_{j=1}^{m} w_j\right) = \theta_2^M w_i\left(\sum_{s=m+1}^{N} w_s\right)$ and

$$\sum_{j=1}^{m}\gamma_{r,j} = -w_r\left[1 + \theta_2^M \sum_{j=1}^{m} w_j\right] = -w_r\left[1 + \theta_2^M\left(1 - \sum_{s=m+1}^{N} w_s\right)\right]$$

(b) Symmetry: $\alpha_{i,j} = \alpha_{j,i}$, $\beta_{r,s} = \beta_{s,r}$, and $\gamma_{r,j} = g_{j,r}$,

(c) Adding-up: $\sum_{i=1}^{m}\alpha_i + \sum_{k=m+1}^{N}\beta_k = 1 - \theta_1^M$,

$$\sum_{i=1}^{m}\alpha_{i,j} = \theta_2^M w_i\left(1 - \sum_{j=1}^{m} w_j\right) = \theta_2^M w_i\left(\sum_{k=m+1}^{N} w_k\right), \text{ and}$$

$$\sum_{i=1}^{m} g_{i,r} = -w_r\left[1 + \theta_2^M \sum_{j=1}^{m} w_j\right] = -w_r\left[1 + \theta_2^M\left(1 - \sum_{s=m+1}^{N} w_s\right)\right]$$

The elasticities can be calculated as follows

(a) Expenditure elasticities: $\varepsilon_i \equiv \dfrac{\alpha_i}{w_i} + \theta_1^M$ and $f_k \equiv \dfrac{\beta_k}{w_k} + \theta_1^M$,

(b) Compensated elasticities: $\varepsilon_{i,j}^c \equiv \dfrac{\alpha_{i,j}}{w_i} + \theta_2^M\left(w_j - \delta_{i,j}\right)$, $f_{k,s}^c \equiv \dfrac{\beta_{k,s}}{w_k} + \theta_2^M\left(w_s - \delta_{k,s}\right)$,

$$p^c_{k,j} \equiv -\frac{\gamma_{k,j}}{w_k} - \theta_2^M w_j \text{, and } q^c_{i,s} \equiv \frac{g_{i,s}}{w_i} + \theta_2^M w_s \,,$$

(c) Uncompensated elasticities:

$$\varepsilon_{i,j} = \left[\frac{\alpha_{i,j}}{w_i} + \theta_2^M\left(w_j - \delta_{i,j}\right)\right] - \left[\frac{\alpha_i}{w_i} + \theta_1^M\right] \cdot \left[w_j + \sum_{k=m+1}^{N} w_k \cdot \left(-\frac{\gamma_{k,j}}{w_k} - \theta_2^M w_j\right)\right]$$

$$f_{k,s} = \left[\frac{\beta_{k,s}}{w_k} + \theta_2^M\left(w_s - \delta_{k,s}\right)\right] - \left[\frac{\beta_k}{w_k} + \theta_1^M\right] \cdot \left[\sum_{r=m+1}^{N} w_r \cdot \left(\frac{\beta_{r,s}}{w_r} + \theta_2^M\left(w_s - \delta_{r,s}\right)\right)\right]$$

$$p_{k,j} = \left[-\frac{\gamma_{k,j}}{w_k} - \theta_2^M w_j\right] - \left[\frac{\beta_k}{w_k} + \theta_1^M\right] \cdot \left[w_j + \sum_{r=m+1}^{N} w_r \cdot \left(-\frac{\gamma_{r,j}}{w_r} - \theta_2^M w_j\right)\right]$$

$$q_{i,s} = \left[\frac{g_{i,s}}{w_i} + \theta_2^M w_s\right] - \left[\frac{\alpha_i}{w_i} + \theta_1^M\right] \cdot \left[\sum_{r=m+1}^{N} w_r \cdot \left(\frac{\beta_{r,s}}{w_r} + \theta_2^M\left(w_s - \delta_{r,s}\right)\right)\right].$$

Table 1. Synthetic Parameters for Three Specifications

Model	Direct		Inverse		Mixed	
	θ_1^D	θ_2^D	θ_1^I	θ_2^I	θ_1^M	θ_2^M
Rotterdam	0	0	0	0	0	0
LA/AIDS	1	1	1	1	1	1
NBR	0	1	0	1	0	1
CBS	1	0	1	0	1	0

1. Restrictions of synthetic parameters to nest popular functional forms for three specifications.
2. Refer to synthetic/generalized demand equation for synthetic coefficients.
For example, synthetic parameters in the direct demand system corresponds to parameters in

$$w_n d\ln q_n = \left[C_n + \theta_1^D w_n\right] d\ln Q + \sum_{n'=1}^{N}\left[C_{n,n'} + \theta_2^D w_n\left(w_{n'} - \delta_{n,n'}\right)\right] d\ln p_{n'}$$

The synthetic parameters for the direct, inverse, and mixed demand functions can be summarized as in Table 1. The value of 0 and 1 for θ_1 captures the constant and variational expenditure or scale coefficients and the value of 0 and 1 for θ_2 represents the constant and variational Slutsky or Antonelli coefficients respectively, where the variations rely on the budget share values. Even though it is difficult to directly compare each of four types of specifications, it is possible to indirectly compare each of them to the synthetic or generalized model, because the synthetic/generalized model nests all four specifications. The joint tests for combinations of possible values of θ_1 and θ_2 can be used to compare among the synthetic/generalized model itself and four nesting differential functional forms within each of direct, inverse, and mixed demand systems respectively.

Relationships among Direct, Inverse, and Mixed Demand Systems

While the statistical tests of the two nesting coefficients can provide the empirical guidance for the best parameterizations among the Rotterdam, LA/AIDS, CBS, NBR, and the generalized functional forms within the direct, inverse, or mixed demand systems, the economic interpretations of the estimated results across three different specifications are not easy. Given one objective of the demand study is to understand and measure the consumers' responsiveness to the changes in exogenous variables, the responsiveness is measured by elasticities or flexibilities, where the elasticity (flexibility) is defined by the percentage change in quantity demanded (willingness to pay) resulting from a 1-percent increase in an exogenous variable. The difficulties arise from the fact that the flexibility (elasticity) matrix has not the simple matrix inversion relation with the elasticity (flexibility) matrix estimated from the direct (inverse) demand functions (e.g., Schultz, 1938, Houck, 1966, and Huang, 1996). Furthermore, the substitutability of the mixed compensated elasticities need not be equivalent to either p-substitutability ($\partial q_n / \partial p_{n'} > 0$) in terms of the direct system, nor q-substitutability ($\partial p_n / \partial q_{n'} < 0$) in terms of the inverse system (Moschini and Vissa, 1993). In this respect, it is worthwhile to derive some functional relationships among the direct, inverse, and mixed demand systems to allow convenient interpretations and comparisons of the estimated results from three alternative specifications.

The relationships among three specifications can be derived based on the mixed demand framework by extending the argument of Moschini and Vissa (1993). While they use a set of identity equations relating the mixed to the direct demand system, there is another set of identity relations relating the mixed to the inverse demand system. Using both sets of identity, we can also derive some relationship between the direct and the inverse demand, based on the mixed demand framework. Following notation is introduced. $E^D_{A,A} \equiv [\varepsilon_{i,i}]$, $E^D_{B,B} \equiv [\varepsilon_{k,s}]$, $E^D_{A,B} \equiv [\varepsilon_{i,k}]$, $E^D_{B,A} \equiv [\varepsilon_{k,i}]$, $E^D_A \equiv [\varepsilon_i]$, and $E^D_B \equiv [\varepsilon_k]$ are the submatrices from the direct demand, $F^I_{A,A} \equiv [f_{i,i}]$, $F^I_{B,B} \equiv [f_{k,s}]$, $F^I_{A,B} \equiv [f_{i,k}]$, $F^I_{B,A} \equiv [f_{k,i}]$, $F^I_A \equiv [f_i]$, and $F^I_B \equiv [f_k]$ are the submatrices from the inverse demand, and $E^M_{A,A} \equiv [\varepsilon_{i,i}]$, $F^M_{B,B} \equiv [f_{k,s}]$, $Q^M_{A,B} \equiv [q_{i,k}]$, $P^M_{B,A} \equiv [p_{k,i}]$, $E^M_A \equiv [\varepsilon_i]$, and $F^M_B \equiv [f_k]$ are the submatrices from the mixed demand. As Moschini and Vissa (1993) demonstrated, the direct demand system is related to the mixed demand system through the identities $q^D_A[p_A, p^M_B(p_A, q_B, y), y] \equiv q^M_A(p_A, q_B, y)$ and $q^D_B[p_A, p^M_B(p_A, q_B, y), y] \equiv \overline{q^M_B}$. By applying a similar logic, the inverse demand system is related to the mixed demand system through the following identities $p^I_A[q^M_A(p_A, q_B, y), q_B, y] \equiv \overline{p_A}$ and $p^I_B[q^M_A(p_A, q_B, y), q_B, y] \equiv p^M_B(p_A, q_B, y)$, which are implied by $\pi^I_A[q^M_A(\pi_A, q_B, 1), q_B, 1] \equiv \overline{\pi_A}$ and $\pi^I_B[q^M_A(\pi_A, q_B, 1), q_B, 1] \equiv \pi^M_B(\pi_A, q_B, 1)$ through the relations of $\pi^I_A[q^M_A(\pi_A, q_B, 1), q_B, 1] \cdot y \equiv \overline{\pi_A} \cdot y$ and $\pi^I_B[q^M_A(\pi_A, q_B, 1), q_B, 1] \cdot y \equiv \pi^M_B(\pi_A, q_B, 1) \cdot y$. From the resulting two kinds of relationships, the other implied relationships can also be derived between the direct and the inverse demand systems. Note that these relationships are based on the partitioning quantity-dependent and price-dependent groups of commodities or the legitimate mixed demand system. Note also that the scale flexibility is defined as responsiveness of (normalized) inverse demand with respect to scale parameter not with respect to expenditure variable. Derivations of following relationships are explained in Appendix B. The resulting relationships among the direct, inverse, and mixed demand functions are summarized as follows:

Theoretical relation of direct elasticities to mixed elasticities. ,

$E_{AA}^{D} = E_{AA}^{M} - Q_{AB}^{M} \cdot \left(F_{BB}^{M}\right)^{-1} P_{BA}^{M}$, $E_{BB}^{D} = \left(F_{BB}^{M}\right)^{-1}$,

$E_{AB}^{D} = Q_{AB}^{M} \cdot \left(F_{BB}^{M}\right)^{-1}$, $E_{BA}^{D} = -\left(F_{BB}^{M}\right)^{-1} P_{BA}^{M}$,

$E_{A}^{D} = E_{A}^{M} - Q_{AB}^{M} \cdot \left(F_{BB}^{M}\right)^{-1} F_{B}^{M}$, $E_{B}^{D} = -\left(F_{BB}^{M}\right)^{-1} F_{B}^{M}$.

Theoretical relation of inverse flexibilities to mixed elasticities.

$F_{AA}^{I} = \left(E_{AA}^{M}\right)^{-1}$, $F_{BB}^{I} = F_{BB}^{M} - P_{BA}^{M}\left(E_{AA}^{M}\right)^{-1} Q_{AB}^{M}$,

$F_{AB}^{I} = -\left(E_{AA}^{M}\right)^{-1} Q_{AB}^{M}$, $F_{BA}^{I} = P_{BA}^{M}\left(E_{AA}^{M}\right)^{-1}$,

$F_{A}^{I} = RowSum\left[\left(E_{AA}^{M}\right)^{-1} \vdots -\left(E_{AA}^{M}\right)^{-1} Q_{AB}^{M}\right]$, $F_{B}^{I} = RowSum\left[P_{BA}^{M}\left(E_{AA}^{M}\right)^{-1} \vdots F_{BB}^{M} - P_{BA}^{M}\left(E_{AA}^{M}\right)^{-1} Q_{AB}^{M}\right]$.

Theoretical relation of mixed elasticities to direct elasticities

$E_{AA}^{M} = E_{AA}^{D} - E_{AB}^{D}\left(E_{BB}^{D}\right)^{-1} E_{BA}^{D}$, $F_{BB}^{M} = \left(E_{BB}^{D}\right)^{-1}$,

$Q_{AB}^{M} = E_{AB}^{D}\left(E_{BB}^{D}\right)^{-1}$, $P_{BA}^{M} = -\left(E_{BB}^{D}\right)^{-1} E_{BA}^{D}$,

$E_{A}^{M} = E_{A}^{D} - E_{AB}^{D}\left(E_{BB}^{D}\right)^{-1} E_{B}^{D}$, $F_{B}^{M} = -\left(E_{BB}^{D}\right)^{-1} E_{B}^{D}$.

Theoretical relations of mixed elasticities to inverse flexibilities

$E_{AA}^{M} = \left(F_{AA}^{I}\right)^{-1}$, $F_{BB}^{M} = F_{BB}^{I} - F_{BA}^{I}\left(F_{AA}^{I}\right)^{-1} F_{AB}^{I}$,

$P_{BA}^{M} = F_{BA}^{I}\left(F_{AA}^{I}\right)^{-1}$, $Q_{AB}^{M} = -\left(F_{AA}^{I}\right)^{-1} F_{AB}^{I}$,

$E_{A}^{M} = -RowSum\left[\left(F_{AA}^{I}\right)^{-1}\right]$, $F_{B}^{M} = I - RowSum\left[F_{BA}^{I}\left(F_{AA}^{I}\right)^{-1}\right]$.

Theoretical relation of direct elasticities to inverse flexibilities

$E_{AA}^{D} = \left(F_{AA}^{I}\right)^{-1} + \left(F_{AA}^{I}\right)^{-1} F_{AB}^{I} \cdot \left[F_{BB}^{I} - F_{BA}^{I}\left(F_{AA}^{I}\right)^{-1} F_{AB}^{I}\right]^{-1} F_{BA}^{I}\left(F_{AA}^{I}\right)^{-1}$, $E_{BB}^{D} = \left[F_{BB}^{I} - F_{BA}^{I}\left(F_{AA}^{I}\right)^{-1} F_{AB}^{I}\right]^{-1}$,

$E_{AB}^{D} = -\left(F_{AA}^{I}\right)^{-1} F_{AB}^{I} \cdot \left[F_{BB}^{I} - F_{BA}^{I}\left(F_{AA}^{I}\right)^{-1} F_{AB}^{I}\right]^{-1}$, $E_{BA}^{D} = -\left[F_{BB}^{I} - F_{BA}^{I}\left(F_{AA}^{I}\right)^{-1} F_{AB}^{I}\right]^{-1} F_{BA}^{I}\left(F_{AA}^{I}\right)^{-1}$,

$$E_A^D = -RowSum\left[\left(F_{AA}^I\right)^{-1} + \left(F_{AA}^I\right)^{-1} F_{AB}^I \cdot \left[F_{BB}^I - F_{BA}^I \left(F_{AA}^I\right)^{-1} F_{AB}^I\right]^{-1} F_{BA}^I \left(F_{AA}^I\right)^{-1} \vdots - \left(F_{AA}^I\right)^{-1} F_{AB}^I \cdot \left[F_{BB}^I - F_{BA}^I \left(F_{AA}^I\right)^{-1} F_{AB}^I\right]^{-1}\right],$$

$$E_B^D = -RowSum\left[-\left[F_{BB}^I - F_{BA}^I \left(F_{AA}^I\right)^{-1} F_{AB}^I\right]^{-1} F_{BA}^I \left(F_{AA}^I\right)^{-1} \vdots \left[F_{BB}^I - F_{BA}^I \left(F_{AA}^I\right)^{-1} F_{AB}^I\right]^{-1}\right].$$

Theoretical relation of inverse flexibilities to direct elasticities

$$F_{AA}^I = \left[E_{AA}^D - E_{AB}^D \left(E_{BB}^D\right)^{-1} E_{BA}^D\right]^{-1}, \qquad F_{BB}^I = \left(E_{BB}^D\right)^{-1} + \left(E_{BB}^D\right)^{-1} E_{BA}^D \left[E_{AA}^D - E_{AB}^D \left(E_{BB}^D\right)^{-1} E_{BA}^D\right]^{-1} E_{AB}^D \left(E_{BB}^D\right)^{-1},$$

$$F_{AB}^I = -\left[E_{AA}^D - E_{AB}^D \left(E_{BB}^D\right)^{-1} E_{BA}^D\right]^{-1} E_{AB}^D \left(E_{BB}^D\right)^{-1}, \quad F_{BA}^I = -\left(E_{BB}^D\right)^{-1} E_{BA}^D \left[E_{AA}^D - E_{AB}^D \left(E_{BB}^D\right)^{-1} E_{BA}^D\right]^{-1},$$

$$F_A^I = RowSum\left[\left[E_{AA}^D - E_{AB}^D \left(E_{BB}^D\right)^{-1} E_{BA}^D\right]^{-1} \vdots - \left[E_{AA}^D - E_{AB}^D \left(E_{BB}^D\right)^{-1} E_{BA}^D\right]^{-1} E_{AB}^D \left(E_{BB}^D\right)^{-1}\right],$$

$$F_B^I = RowSum\left[-\left(E_{BB}^D\right)^{-1} E_{BA}^D \left[E_{AA}^D - E_{AB}^D \left(E_{BB}^D\right)^{-1} E_{BA}^D\right]^{-1} \vdots \left(E_{BB}^D\right)^{-1} + \left(E_{BB}^D\right)^{-1} E_{BA}^D \left[E_{AA}^D - E_{AB}^D \left(E_{BB}^D\right)^{-1} E_{BA}^D\right]^{-1} E_{AB}^D \left(E_{BB}^D\right)^{-1}\right].$$

The identified relationships among three specifications allow retrieving the usual elasticity form in the direct demand system from the estimates of the mixed as well as the inverse demand systems. On the other hand, the overall evaluations of three alternative specifications are still not easy. The difficulties to compare different specifications across direct, inverse, and mixed demand systems are that the alternative specifications are non-nested relative to each other and non-nested hypotheses testing approach oftentimes does not provide definite answer for this problem. Unlike the non-nesting test procedures and artificial nesting approach, the model selection criterion does not require actual estimation of the composite model. In this respect, the model selection approach, such as the Likelihood Dominance Criterion introduced by Pollak and Wales (1991), provides an alternative method to rank competing models as long as the competing specifications have the common dependent variables. Furthermore, Saha, Shumway, and Talpaz (1994) demonstrated that the likelihood dominance criterion outperformed some widely used non-nested testing procedures such as Davidson-MacKinnon J test and Cox test in selecting the true model, using Monte Carlo evidence.

Let H_1 and H_2 denote two non-nesting hypotheses and n_1, n_2 and L_1, L_2 are the corresponding number of independent parameters and log-likelihood values with assumption of

$n_1 \le n_2$. Let $C(v,\tau)$ denote the critical values of the chi-square distribution with v degrees-of-freedom at some fixed significant level τ. Pollak and Wales (1991) demonstrate that the use of three model selection rules can result in one of three possible outcomes:

(a) H_2 is preferred to H_1,

iff $(1/2)\cdot[C(n_2-n_1+1,\tau)-C(1,\tau)]<L_2-L_1$ or $L_1<L_2$ for $n_1=n_2$

or $(1/2)\cdot(n_2-n_1)<L_2-L_1$ for likelihood dominance criterion of $n_C\to\infty$

or $(\log T/2)\cdot(n_2-n_1)<L_2-L_1$ for Schwarz model selection rule

or $(n_2-n_1)<L_2-L_1$ for Akaike model selection rule.

(b) H_1 is preferred to H_2,

iff $L_2-L_1<(1/2)\cdot[C(n_2+1,\tau)-C(n_1+1,\tau)]$ or $L_2<L_1$ for $n_1=n_2$

or $L_2-L_1<(1/2)\cdot(n_2-n_1)$ for likelihood dominance criterion of $n_C\to\infty$

or $L_2-L_1<(\log T/2)\cdot(n_2-n_1)$ for Schwarz model selection rule

or $L_2-L_1<(n_2-n_1)$ for Akaike model selection rule.

(c) Indecisive between H_1 and H_2,

iff $(1/2)\cdot[C(n_2+1,\tau)-C(n_2+1,\tau)]<L_2-L_1<(1/2)\cdot[C(n_2-n_1+1,\tau)-C(1,\tau)]$

or $L_1=L_2$ for $n_1=n_2$

or $L_2-L_1=(1/2)\cdot(n_2-n_1)$ for likelihood dominance criterion of $n_C\to\infty$

or $L_2-L_1=(\log T/2)\cdot(n_2-n_1)$ for Schwarz model selection rule

or $L_2-L_1=(n_2-n_1)$ for Akaike model selection rule.

The similar implications of the Likelihood Dominance and the two common model selection criteria of Akaike Information criterion (Akaike, 1973) and Schwarz information criterion (Schwarz, 1978) can be understood as follows: The Akaike and Schwarz model selection rules of choosing the largest value of $L_i - n_i$ and $L_i - (\log T/2)\cdot n_i$ can be understood as pair-wise comparison rules for $L_2 - L_1$ in terms of relative penalty functions $(n_2 - n_1)$ and $(\log T/2)\cdot(n_2 - n_1)$ respectively. These two relative penalty functions have similar implications as the likelihood dominance criterion, as Pollak and Wales (1991) argued that $(1/2)\cdot[C(n_C - n_1, \tau) - C(n_C - n_2, \tau)]$ converges to $(1/2)\cdot(n_2 - n_1)$ as $n_C \to \infty$ based on the asymptotic normality property as a function of degrees-of-freedom of the chi-squared distribution.

Note that non-nesting hypotheses hypothesis should involve the same dependent variables for the above discussions. If the hypotheses involve different dependent variables but are functionally related, then the likelihood function must be adjusted by including the appropriate Jacobian bias term. To avoid the difficulties involved this adjustment, the synthetic models with the different Rotterdam-type dependent variables are further transformed into the generalized models with the common AIDS-type dependent variables across three alternative specifications. In this respect, the generalized functional forms allow the more convenient comparisons based on the model selection approaches. Note also that to narrow this indecisive range, the significant level τ be adjustably selected and/or the composite parametric size n_C can be determined directly from the significance tables for the chi-square distribution for given n_1, n_2 and L_1, L_2.

In the next section, the following empirical procedure is applied to study consumer behavior for soft drink consumptions: (i) the graphical causal methods of the PC and GES algorithms are applied and the information of local causal structure is used to obtain an empirical guidance for the specification choice among three demand specifications; (ii) the generalized functional forms for three specifications are estimated and the statistical tests of synthetic coefficients are used to choose the best parameterizations among the Rotterdam, LA/AIDS, CBS, NBR, and generalized functional forms within each of the three specification; (iii) the estimated results of the chosen parameterization are interpreted and compared based on the identified relationship among three demand specifications. Finally, the overall results of three alternative specifications are evaluated based on the model selection frameworks.

III. Empirical Results

Data Description

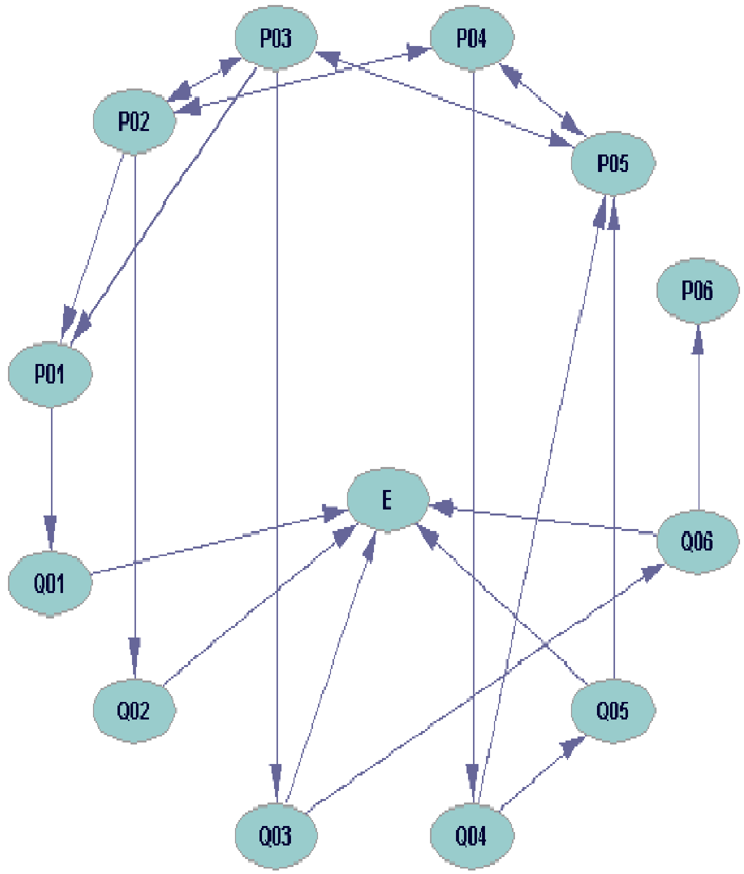
The data set consists of weekly observations on 23 soft drink products with size of 6/12 oz sold at Dominick's Finer Foods (DFF) from 09:14:1989 through 09:22:1993 with the sample size 210. All the data are from the Dominick's database, which is publicly available from the University of Chicago Graduate School of Business (http://www.chicagogsb.edu/). The original data set is the store level weekly retail scanner data for the specific items represented by UPC code. The Dominick's Finer Foods (DFF) is the second largest supermarket chain in the Chicago metropolitan area with about 25% market share. Each soft drink used for this study is a specific soft drink of 6/12 oz size such as Coca-cola classic, Pepsi-cola cans, Seven-up diet can.

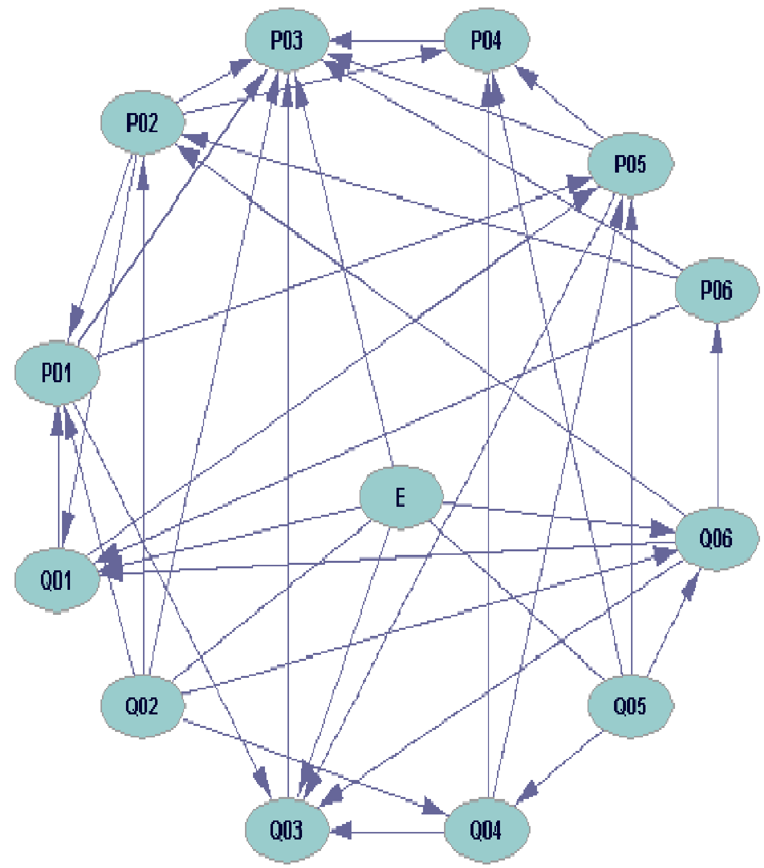
The chain level data for the aggregated commodity groups is used for this study. In order to characterize the chain level characteristics, the store level data are aggregated across stores by using the simple sum and unit value for quantity and price variables, where unit value is total sale revenue divided by the total quantity sold. For commodity aggregation, the 23 soft drink products are aggregated as following 6 groups: Coca-Cola and Sprite (group 01), Pepsi-Cola and Mountain Dew (group 02), Seven-Up and Dr Pepper (group 03), Lipton Brisk (group 04), A&W and Rite-Cola (group 05), and Sunkist and Canada Dry (group 06). The choice of 6/12 oz size and the commodity grouping are based on the data availability and identified homogeneity in terms of the co-movements of price and quantity variables in the preliminary study. The Tornqvist-Theil indexes are used to represent price and quantity variables for each group.

For the purpose of estimating differential demand systems, the differential terms for price and quantity variables are approximated by the finite first differences ($d\ln p_n \approx \ln p_{n,t} - \ln p_{n,t-1}$ and $d\ln q_n \approx \ln q_{n,t} - \ln q_{n,t-1}$) and the market share terms are replaced by their moving average ($w_n \approx \left(w_{n,t} + w_{n,t-1}\right)/2$). The market share changes dw are approximated based on the log differential property ($dw = w \cdot d\ln w \approx (1/2) \cdot \left(w_{n,t} + w_{n,t-1}\right) \cdot \left(\ln w_{n,t} - \ln w_{n,t-1}\right)$), since dw has a limited range of $(-1, 1)$, whereas $dw = w \cdot d\ln w$ has a range of $(-\infty, \infty)$ (Barten, 1993). The preliminary unit root tests imply that these transformed variables in differential demand system are all stationary. These results are consistent with the observation in the demand literature that the differential demand system has been considered as appropriate specification to deal with the possible non-stationarity problems.

Local Causal Structures among Prices and Quantities

The specification choice is closely related with the identification issue of the local causal structure between price and quantity for a specific commodity. We apply the graphical causal models of the PC and GES algorithms to inductively derive this local causal structure. The empirical results are presented in Figure 1 and 2. There remain several undecided causal directions in both results and such directions cannot be decided without additional causal information. The undirected edges in the result of the GES algorithm represent the limitations to identify causal directions based on the statistical observations only (observational equivalence). On the other hand, the bi-directed edges in the result of PC algorithm imply the existence of unobserved factors. The capability of identifying unobserved factors between two variables,





1. P and Q denotes representative price and quantity indices for each group defended as
   Group 01: Coca-Cola and Sprite, Group 02: Pepsi-Cola and Mountain Dew, Group 03: Seven-Up and Dr Pepper,
   Group 04: Lipton Brisk., Group 05: A&W and Rite-Cola, Group 06: Sunkist and Canada Dry, and E denote total expenditure variable.
2. The result of PC algorithm is based on the significant level of 0.1, which is recommended for sample size of 100-300 (Spirtes et al., 2000).

Figure 1. Causal Structure Inferred by PC Algorithm

Figure 2. Causal Structure Inferred by GES Algorithm

based on the tetrad relationship among partial correlations, is one advantage of the PC algorithm relative to the GES algorithm. On the other hand, given the Markov condition (causal sufficiency and acyclic assumptions), the GES algorithm has following advantages relative to the PC algorithm (i) The GES algorithm does not require the choice of the significant level. This is advantage, given that the result of PC algorithm oftentimes is sensitive to the choice of the significant level. (ii) The GES algorithm oftentimes provides finer results than the PC algorithm. The difference is due to the fact that the GES algorithm is based on the numerical scores on the overall hypothetic causal structures, whereas the PC algorithm is based on the categorical decision on individual edges and such categorical decisions can be sensitive to the chosen significant level. In our results, the GES algorithm provides all the edges (skeleton) identified by the PC algorithm with some additional edges. Sometimes these additional edges are important to decide the causal directions among variables. For example, the edge $P01-Q02$ is crucial to orient $Q01 \rightarrow P01$ in the GES algorithm, because this orientation is based on the unshielded collider pattern of $Q01 \rightarrow P01 \leftarrow Q02$. In the PC algorithm, the edge $P01-Q02$ is statistically removed and this categorical decision can be sensitive to the specified significant level. Similar patterns such as $P02-P06$ for $Q02 \rightarrow P02 \leftarrow P06$ and $Q02-P03$ for $Q02 \rightarrow P03 \leftarrow Q03$ can be used to explain the different implications for local causal structure between price and quantity between PC and GES algorithms. In this respect, the results of the PC algorithm need to be carefully used for the choice of the significant level. In fact, the local causal structures between price and quantity variables inferred by the PC algorithm are sensitive to the change of the significant level. In this study, the final result of PC algorithm is based on the significant level of 0.1, which is recommended for sample size of 100-300 (Spirtes et al., 2000).

For the full use of theoretical information from the demand theory, all we need is the local causal structures between price and quantity variables for each commodity. This local information provides the data-based information to address the choice issue among three possible specifications of direct, inverse, and mixed demand functions. The local causal structures identified by the PC algorithm imply the mixed demand system, where quantity dependent specifications are suggested for commodity groups of 01 (Coca-Cola and Sprite), 02 (Pepsi-Cola and Mountain Dew), 03 (Seven-Up and Dr Pepper), and 04 (Lipton Brisk) and price dependent specifications are suggested for commodity groups of 05 (A&W and Rite-Cola) and 06 (Sunkist and Canada Dry). On the other hand, the local causal structures identified by the GES algorithm imply the inverse demand system, where price dependent specifications are suggested for all the aggregate commodities. Given that the direct demand system or quantity dependent specification is widely used in empirical studies, the possibility of the price dependent or the mixed demand specification implied from the GES and PC algorithms need to be interpreted. One possible interpretation is that (i) The soft drinks are differentiated products, where the differentiated products are defined as the products differentiated by taste, packing and brand-base advertisement to influence consumers' perception of different brands, and (ii) The retail prices for differentiated products can be determined by strategic pricing rules of firms incorporating supply and demand characteristics for these products (Dhar, Chavas, and Gould, 2003).

Estimations of Direct, Inverse, and Mixed Demand Systems

The generalized functional forms for the inverse and the mixed demand systems as well as the direct demand specification are estimated to study the consumption pattern for the soft drinks. The direct demand system is estimated for the comparison purpose with the inverse and mixed demand systems, which are chosen based on the local causal structure of the GES and PC algorithms respectively. The estimated parameters in all three direct, inverse, and mixed synthetic demand systems of the common differential AIDS type dependent variable are presented in Table 2. All three types of demand systems are estimated by the nonlinear seemingly unrelated regression estimation method with allowing autoregressive errors (SHAZAM). The first order autocorrelation is used with the restriction that the autocorrelation coefficients are constrained to be the same in all equations. The homogeneity, symmetry, and adding-up properties are used for the economy of parameters in empirical models. One equation is dropped in estimation step for the direct and inverse demand, since the adding-up condition in direct or inverse demand makes the demand system singular. The parameters in the dropped equation are recovered by using the homogeneity, symmetry, and adding-up conditions. On the other hand, all the equations are used in estimation for the mixed demand, since the adding-up condition holds only at a point and thus does not induce the singularity in the resulting system. The number of independent parameters in all the demand systems is 23 for all three demand specifications, which include the two synthetic parameters and one autocorrelation correction term.

For the comparison of different parameterization assumptions of the constant and/or variation for the expenditure (scale) coefficient and Slutsky (Antonelli) coefficient within each of direct, inverse, and mixed demand system, the Wald statistic, which is distributed chi-square with the same degrees of freedom as the number of restrictions, is used. The empirical results of these comparison statistics are presented in Table 3. Within each of direct, inverse, and mixed demand system, all the nested Rotterdam, LA/AIDS, NBR, and CBS specifications, which assume the fixed restriction on the synthetic parameters, are strongly rejected. This test results imply that none of the four nested models is adequate and the generalized functional forms are

Table 2. Parameter Estimates

	Direct Model					Inverse Model					Mixed Model			
Coefficient	Estimate	Std. Error	t-Statistic	p-value	Coefficient	Estimate	Std. Error	t-Statistic	p-value	Coefficient	Estimate	Std. Error	t-Statistic	p-value
th1	1.3852	0.0338	41.0025	0.0000	th1	0.9609	0.0084	113.9911	0.0000	th1	0.1086	0.0502	2.1641	0.0305
th2	4.7255	0.1193	39.6028	0.0000	th2	0.1852	0.0068	27.0705	0.0000	th2	-0.1618	0.0464	-3.4893	0.0005
c01	-0.1119	0.0110	-10.2124	0.0000	d01	-0.0144	0.0027	-5.3288	0.0000	a01	0.2790	0.0183	15.2047	0.0000
c02	-0.0813	0.0114	-7.1276	0.0000	d02	-0.0102	0.0030	-3.4277	0.0006	a02	0.3470	0.0200	17.3620	0.0000
c03	-0.0771	0.0086	-8.9905	0.0000	d03	-0.0104	0.0023	-4.5423	0.0000	a03	0.2233	0.0165	13.5553	0.0000
c04	-0.0363	0.0021	-17.0796	0.0000	d04	-0.0072	0.0007	-10.2280	0.0000	a04	0.0280	0.0060	4.6852	0.0000
c05	-0.0700	0.0070	-9.9813	0.0000	d05	-0.0085	0.0019	-4.5498	0.0000	b05	-0.0010	0.0047	-0.2020	0.8399
c06*	-0.0086	0.0071	-1.2171	0.2236	d06*	0.0116	0.0041	2.8590	0.0043	b06*	0.0150	0.0031	4.9234	0.0000
c11	0.1552	0.0486	3.1933	0.0014	d11	-0.0046	0.0024	-1.9450	0.0518	a11	-1.1976	0.0683	-17.5455	0.0000
c12	0.0393	0.0319	1.2314	0.2182	d12	-0.0019	0.0013	-1.4661	0.1426	a12	0.6802	0.0566	12.0154	0.0000
c13	-0.0851	0.0289	-2.9473	0.0032	d13	0.0002	0.0013	0.1935	0.8465	a13	0.4324	0.0504	8.5805	0.0000
c14	-0.0083	0.0108	-0.7693	0.4417	d14	0.0002	0.0005	0.4582	0.6468	a14*	0.0759	0.0259	2.9288	0.0034
c15	-0.0626	0.0241	-2.5933	0.0095	d15	-0.0003	0.0012	-0.2691	0.7879	a22	-1.2570	0.0726	-17.3187	0.0000
c16*	-0.0385	0.0209	-1.8430	0.0653	d16*	0.0064	0.0012	5.5284	0.0000	a23	0.4667	0.0583	8.0043	0.0000
c22	0.0690	0.0466	1.4810	0.1386	d22	-0.0019	0.0024	-0.8231	0.4105	a24*	0.1007	0.0279	3.6095	0.0003
c23	-0.0027	0.0276	-0.0965	0.9232	d23	-0.0034	0.0012	-2.8513	0.0044	a33	-0.9733	0.0751	-12.9635	0.0000
c24	-0.0375	0.0111	-3.3800	0.0007	d24	0.0006	0.0005	1.2309	0.2183	a34*	0.0678	0.0232	2.9224	0.0035
c25	-0.0464	0.0256	-1.8114	0.0701	d25	-0.0005	0.0012	-0.3980	0.6906	a44*	-0.2708	0.0253	-10.7210	0.0000
c26*	-0.0218	0.0208	-1.0484	0.2944	d26*	0.0071	0.0013	5.4636	0.0000	b55	-0.0374	0.0075	-5.0124	0.0000
c33	0.1427	0.0435	3.2816	0.0010	d33	-0.0003	0.0022	-0.1390	0.8894	b56	0.0071	0.0021	3.4392	0.0006
c34	-0.0133	0.0095	-1.3949	0.1631	d34	0.0005	0.0004	1.0416	0.2976	b66	-0.0383	0.0066	-5.8044	0.0000
c35	-0.0224	0.0209	-1.0699	0.2847	d35	-0.0025	0.0011	-2.2266	0.0260	r51	-0.0079	0.0085	-0.9321	0.3513
c36*	-0.0192	0.0215	-0.8928	0.3720	d36*	0.0055	0.0011	4.8862	0.0000	r52	-0.0393	0.0103	-3.7990	0.0002
c44	0.1097	0.0156	7.0345	0.0000	d44	-0.0052	0.0010	-4.9888	0.0000	r53	-0.0331	0.0107	-3.0846	0.0020
c45	-0.0322	0.0132	-2.4347	0.0149	d45	0.0017	0.0007	2.5608	0.0104	r54*	-0.0156	0.0056	-2.7905	0.0053
c46*	-0.0183	0.0065	-2.8149	0.0049	d46*	0.0022	0.0004	5.5577	0.0000	r61	-0.0137	0.0077	-1.7791	0.0752
c55	0.1622	0.0391	4.1501	0.0000	d55	0.0005	0.0018	0.2620	0.7933	r62	-0.0215	0.0085	-2.5286	0.0115
c56*	0.0014	0.0224	0.0644	0.9487	d56*	0.0011	0.0011	0.9812	0.3265	r63	-0.0394	0.0092	-4.2923	0.0000
c66*	0.0964	0.0340	2.8321	0.0046	d66*	-0.0224	0.0035	-6.4711	0.0000	r64*	-0.0092	0.0035	-2.6017	0.0093
rho	-0.3569	0.0303	-11.7773	0.0000	rho	-0.3614	0.0296	-12.2266	0.0000	rho	-0.3660	0.0278	-13.1655	0.0000

1. Each number represent each group defended as Group01: Coca-Cola and Sprite, Group02: Pepsi-Cola and Mountain Dew, Group03: Seven-Up and Dr Pepper, Group04: Lipton Brisk., Group05: A&W and Rite-Cola, and Group06: Sunkist and Canada Dry. For example, c12 corresponds to parameter in quantity equation of group01 w.r.t. group02 price variable in
$$dw_n = \left[C_n - \left(1-\theta_1^O\right)w_n\right]d\ln Q + \sum_{n'=1}^{N}\left[C_{n,n'} - \left(1-\theta_2^O\right)w_n\left(w_{n'} - \delta_{n,n'}\right)\right]d\ln p_{n'}$$
2. Coefficients with * mark are derived based on the adding-up and homogeneity conditions.

the statistically better specification for all three demand specifications. In this respect, the generalized functional form of the common differential AIDS type dependent variable is used for the comparison across the direct, inverse, and mixed demand system.

Table 3. Comparison Statistics for Three Specifications

Restrictions on Synthetic parameters	Direct Wald statistic	Direct p-value	Inverse Wald statistic	Inverse p-value	Mixed Wald statistic	Mixed p-value
th1 = 0	1681.2049	0.0000	12993.9780	0.0000	4.6833	0.0305
th2 = 0	1568.3829	0.0000	732.8099	0.0000	12.1754	0.0005
th1 = 1	129.9852	0.0000	21.5216	0.0000	315.5424	0.0000
th2 = 1	974.8223	0.0000	14180.2140	0.0000	628.0337	0.0000
th1 = 0 & th2 = 0	3032.4904	0.0000	13000.9610	0.0000	12.6597	0.0018
th1 = 1 & th2 = 1	1059.2406	0.0000	14640.0880	0.0000	3708.4420	0.0000
th1 = 0 & th2 = 1	2485.3570	0.0000	34603.8330	0.0000	1267.3297	0.0000
th1 = 1 & th2 = 0	1642.1024	0.0000	847.4041	0.0000	967.7887	0.0000

1. Refer to synthetic/generalized demand equation for synthetic coefficients. For example, th1 and th2 corresponds the synthetic parameters in the direct demand system of $\theta_1^D and\, \theta_2^D$ in $w_n d\ln q_n = \left[C_n + \theta_1^D w_n\right] d\ln Q + \sum_{n'=1}^{N}\left[C_{n,n'} + \theta_2^D w_n\left(w_{n'} - \delta_{n,n'}\right)\right] d\ln p_{n'}$

The compensated and uncompensated elasticities/flexibilities estimates with their standard errors and corresponding p-values for the direct, inverse, and mixed demand systems are presented in Table 4. In the results of the direct demand system, the own elasticities are all negative and statistically significant. The expenditure elasticities are close to unity, as expected for the normal goods. The soft drinks are net and gross p-substitutes for each other, given that negative estimates $\varepsilon_{4,5}^{c,D}$, $\varepsilon_{5,4}^{c,D}$, $\varepsilon_{4,5}^{D}$, $\varepsilon_{5,4}^{D}$, and $\varepsilon_{6,4}^{D}$ are insignificant, where $\varepsilon_{n,n'}^{c,D}$ and $\varepsilon_{n,n'}^{D}$ denote the compensated and uncompensated elasticities in the direct demand system. In the results of the inverse demand system, the own flexibilities are all negative and statistically significant. The scale flexibilities are close to unity in absolute values, as expected for the normal goods. The soft drinks are gross q-substitutes for each other. Note that the compensated flexibilities in inverse

Table 4. Elasticities/Flexibilities Estimates

	Direct Compensated						
	P01	P02	P03	P04	P05	P06	note
Q01	-2.871	1.468	0.596	0.193	0.289	0.313	Group01
	0.149	0.112	0.104	0.040	0.087	0.075	Coke
	0.000	0.000	0.000	0.000	0.001	0.000	Sprite
Q02	1.424	-3.156	0.900	0.090	0.354	0.376	Group02
	0.109	0.134	0.094	0.039	0.091	0.072	Pepsi
	0.000	0.000	0.000	0.020	0.000	0.000	Mt. Dew
Q03	0.841	1.310	-3.075	0.155	0.403	0.354	Group03
	0.147	0.137	0.194	0.049	0.108	0.110	7-up
	0.000	0.000	0.000	0.002	0.000	0.001	Dr Pepper
Q04	1.108	0.533	0.629	-2.189	-0.160	0.068	Group04
	0.228	0.228	0.200	0.266	0.275	0.135	Lipton
	0.000	0.020	0.002	0.000	0.561	0.616	Brisk
Q05	0.714	0.902	0.705	-0.069	-2.731	0.467	Group05
	0.214	0.232	0.189	0.118	0.324	0.203	A&W
	0.001	0.000	0.000	0.561	0.000	0.022	Rite Cola
Q06	0.883	1.097	0.709	0.034	0.535	-3.269	Group06
	0.212	0.211	0.220	0.067	0.233	0.313	Sunkist
	0.000	0.000	0.001	0.616	0.022	0.000	Canada Dry

	Inverse Compensated						
	Q01	Q02	Q03	Q04	Q05	Q06	Group
P01	-0.152	0.045	0.037	0.010	0.019	0.041	Group01
	0.006	0.004	0.004	0.002	0.004	0.004	Coke
	0.000	0.000	0.000	0.000	0.000	0.000	Sprite
P02	0.043	-0.140	0.023	0.011	0.019	0.043	Group02
	0.004	0.006	0.004	0.002	0.004	0.005	Pepsi
	0.000	0.000	0.000	0.000	0.000	0.000	Mt. Dew
P03	0.052	0.034	-0.151	0.011	0.007	0.046	Group03
	0.006	0.006	0.009	0.002	0.006	0.006	7-up
	0.000	0.000	0.000	0.000	0.198	0.000	Dr Pepper
P04	0.055	0.065	0.045	-0.286	0.056	0.064	Group04
	0.011	0.010	0.009	0.018	0.014	0.008	Lipton
	0.000	0.000	0.000	0.000	0.000	0.000	Brisk
P05	0.047	0.048	0.013	0.024	-0.161	0.028	Group05
	0.010	0.010	0.010	0.006	0.014	0.010	A&W
	0.000	0.000	0.198	0.000	0.000	0.006	Rite Cola
P06	0.117	0.126	0.093	0.032	0.032	-0.400	Group06
	0.012	0.013	0.011	0.004	0.012	0.034	Sunkist
	0.000	0.000	0.000	0.000	0.006	0.000	Canada Dry

	Mixed Compensated						
	P01	P02	P03	P04	Q05	Q06	Group
Q01	-4.291	2.459	1.561	0.272	-0.047	-0.066	Group01
	0.257	0.210	0.186	0.095	0.031	0.028	Coke
	0.000	0.000	0.000	0.004	0.134	0.018	Sprite
Q02	2.385	-4.372	1.635	0.352	-0.158	-0.092	Group02
	0.204	0.264	0.208	0.100	0.037	0.030	Pepsi
	0.000	0.000	0.000	0.000	0.000	0.002	Mt. Dew
Q03	2.203	2.380	-4.928	0.345	-0.190	-0.220	Group03
	0.263	0.303	0.394	0.121	0.055	0.048	7-up
	0.000	0.000	0.000	0.004	0.001	0.000	Dr Pepper
Q04	1.555	2.078	1.399	-5.555	-0.347	-0.209	Group04
	0.546	0.589	0.490	0.526	0.118	0.074	Lipton
	0.004	0.000	0.004	0.000	0.003	0.005	Brisk
P05	0.028	0.312	0.270	0.135	-0.196	0.049	Group05
	0.079	0.097	0.100	0.051	0.035	0.017	A&W
	0.725	0.001	0.007	0.008	0.000	0.003	Rite Cola
P06	0.098	0.179	0.378	0.088	0.056	-0.252	Group06
	0.083	0.092	0.096	0.037	0.019	0.032	Sunkist
	0.236	0.052	0.000	0.018	0.003	0.000	Canada Dry

	Direct Uncompensated						
	P01	P02	P03	P04	P05	P06	Expenditure
Q01	-3.135	1.196	0.409	0.147	0.182	0.219	0.973
	0.150	0.111	0.104	0.040	0.087	0.075	0.020
	0.000	0.000	0.000	0.000	0.036	0.004	0.000
Q02	1.126	-3.462	0.689	0.038	0.234	0.271	1.095
	0.109	0.133	0.094	0.039	0.091	0.073	0.018
	0.000	0.000	0.000	0.322	0.011	0.000	0.000
Q03	0.574	1.034	-3.264	0.108	0.295	0.260	0.985
	0.146	0.137	0.193	0.050	0.109	0.110	0.028
	0.000	0.000	0.000	0.029	0.007	0.018	0.000
Q04	0.939	0.359	0.509	-2.219	-0.228	0.008	0.621
	0.228	0.227	0.200	0.266	0.275	0.136	0.035
	0.000	0.114	0.011	0.000	0.407	0.952	0.000
Q05	0.511	0.692	0.561	-0.104	-2.814	0.395	0.749
	0.213	0.231	0.189	0.119	0.325	0.204	0.055
	0.017	0.003	0.003	0.378	0.000	0.053	0.000
Q06	0.532	0.734	0.460	-0.028	0.392	-3.393	1.295
	0.211	0.210	0.220	0.067	0.233	0.314	0.069
	0.012	0.000	0.036	0.677	0.092	0.000	0.000

	Inverse Uncompensated						
	Q01	Q02	Q03	Q04	Q05	Q06	Scale
P01	-0.427	-0.239	-0.159	-0.038	-0.092	-0.056	-1.014
	0.006	0.004	0.004	0.002	0.004	0.004	0.006
	0.000	0.000	0.000	0.000	0.000	0.000	0.000
P02	-0.227	-0.420	-0.168	-0.036	-0.091	-0.053	-0.997
	0.004	0.005	0.004	0.002	0.004	0.005	0.006
	0.000	0.000	0.000	0.000	0.000	0.000	0.000
P03	-0.224	-0.250	-0.347	-0.037	-0.104	-0.051	-1.015
	0.006	0.005	0.009	0.002	0.006	0.006	0.008
	0.000	0.000	0.000	0.000	0.000	0.000	0.000
P04	-0.247	-0.247	-0.169	-0.339	-0.066	-0.043	-1.113
	0.010	0.009	0.009	0.019	0.014	0.008	0.013
	0.000	0.000	0.000	0.000	0.000	0.000	0.000
P05	-0.234	-0.243	-0.187	-0.025	-0.275	-0.072	-1.038
	0.011	0.009	0.010	0.006	0.014	0.010	0.015
	0.000	0.000	0.000	0.000	0.000	0.000	0.000
P06	-0.111	-0.109	-0.069	-0.008	-0.061	-0.481	-0.840
	0.015	0.016	0.013	0.004	0.012	0.035	0.041
	0.000	0.000	0.000	0.057	0.000	0.000	0.000

	Mixed Uncompensated						
	P01	P02	P03	P04	Q05	Q06	Expenditure
Q01	-4.614	2.083	1.267	0.191	-0.028	-0.044	1.136
	0.260	0.208	0.186	0.096	0.031	0.028	0.038
	0.000	0.000	0.000	0.046	0.363	0.110	0.000
Q02	2.002	-4.819	1.287	0.257	-0.136	-0.067	1.348
	0.203	0.263	0.209	0.097	0.036	0.030	0.042
	0.000	0.000	0.000	0.008	0.000	0.025	0.000
Q03	1.843	1.959	-5.256	0.255	-0.169	-0.196	1.269
	0.263	0.302	0.398	0.121	0.055	0.048	0.066
	0.000	0.000	0.000	0.035	0.002	0.000	0.000
Q04	1.357	1.846	1.218	-5.604	-0.336	-0.196	0.699
	0.548	0.578	0.487	0.533	0.118	0.074	0.113
	0.013	0.001	0.012	0.000	0.004	0.008	0.000
P05	-0.001	0.278	0.244	0.127	-0.194	0.051	0.100
	0.079	0.095	0.100	0.051	0.034	0.017	0.028
	0.993	0.004	0.015	0.012	0.000	0.002	0.000
P06	0.023	0.091	0.310	0.069	0.060	-0.247	0.265
	0.085	0.094	0.097	0.037	0.019	0.031	0.045
	0.787	0.331	0.001	0.058	0.002	0.000	0.000

* P and Q denotes representative price and quantity indices for each group defended as Group 01: Coca-Cola and Sprite, Group 02: Pepsi-Cola and Mountain Dew, Group 03: Seven-Up and Dr Pepper, Group 04: Lipton Brisk., Group 05: A&W and Rite-Cola, Group 06: Sunkist and Canada Dry, and E denote total expenditure variable

* In each cell, the first element is the estimates, the second is the standard error, and the third is the associated p-value.

demand system are imperfect measures of the interaction of goods in their satisfaction of wants, since the dominating complementarity $f^{c}_{n,n'} > 0$ does not come from the preference structures but from the adding-up or homogeneity condition $\sum_{n'=1}^{N} f^{c}_{n,n'} = 0$ together with the negativity condition $f^{c}_{n,n'} < 0$ (Barten and Bettendorf, 1989). Note that the magnitudes of the compensated cross flexibilities are relatively small. In the results of the mixed demand system, the own elasticities and/or flexibilities are all negative and statistically significant. The expenditure elasticities are close to unity, as expected for the normal goods. The soft drinks are net and gross substitutes each other, given that negative estimate $p^{M}_{5,1}$ is insignificant. The exceptions are $f^{c,M}_{5,6}$, $f^{c,M}_{6,5}$, $f^{M}_{5,6}$, and $f^{M}_{6,5}$, whose magnitudes are relatively small compared to other estimates.

Note that the substitutability of the mixed compensated elasticities need not be equivalent to either p-substitutability in terms of the direct system, nor q-substitutability in terms of the inverse system, where the $\partial q_{n} / \partial p_{n'} > 0$ means p-substitutability in terms of the direct system and the $\partial p_{n} / \partial q_{n'} < 0$ q- substitutability in terms of the inverse system (Moschini and Vissa, 1993). Note also that the expenditure elasticities for quantity dependent group (group 01-04) measure percentage changes in consumption with respect to one percent increase in total expenditure as in the direct demand system, whereas the expenditure elasticities for price dependent group (group 05-06) measure percentage changes in willingness to pay with respect to one percent increase in total expenditure. On the other hand, the scale flexibilities measure percentage changes in normalized price with respect to one percent increase in the proportionate increase in consumption. For example, for group 05 (A&W and Rite Cola), the percentage increase in consumption with respect to one percent increase in total expenditure is 0.749 estimated in the direct demand system, the percentage increase in willingness to pay with respect to one percent increase in total expenditure is 0.100 estimated in the mixed demand system, and the percentage decrease in normalized price with respect to one percent increase in the proportionate increase in consumption is 1.038 estimated in the inverse demand system.

Comparisons of Direct, Inverse, and Mixed Demand Systems

The convenient and familiar forms of comparison are possible across the direct, inverse, and mixed demand systems in terms of one of three possible forms: the elasticities in the form of direct demand system, the flexibilities in the form of inverse demand system, and the elasticities in the form of mixed demand system. These results are retrieved based on the derived relationships among the direct, inverse, and mixed demand systems. The relationships across the direct, inverse, and mixed demand system in terms of the uncompensated elasticities/flexibilities retrieved from the direct, inverse, and mixed demand system are presented in Table 5. The tables in diagonal positions are replicated from the estimated ones and the own and expenditure/scale elasticities/flexibilities are summarized in the tables at the bottom positions.

The own elasticities and/or flexibilities are all negative and the soft drinks are gross substitutes each other, given that the insignificance estimates imply the insignificant corresponding retrieved ones. For example, the insignificant estimate $\varepsilon_{5,4}^{D}$ in the direct demand system implies the corresponding insignificant retrieved one $p_{5,4}^{M}$ in the mixed demand form retrieved from the direct system estimates. Overall, the expenditure elasticities and scale flexibilities are close to unity, as expected for the normal goods. Recall that the expenditure elasticities for the direct demand system and for the quantity dependent variables group in the mixed demand system, the expenditure elasticities for the price dependent variables group in the mixed demand system, and the scale flexibility for the inverse demand system measure different responses of consumers with respect to the changes in different variables as discussed.

Table 5. Elasticities/Flexibilities Comparisons

Direct Form Estimated from Direct Model

	P01	P02	P03	P04	P05	P06	Expenditure
Q01	-3.135	1.196	0.409	0.147	0.182	0.219	0.973
Q02	1.126	-3.462	0.689	0.038	0.234	0.271	1.095
Q03	0.574	1.034	-3.264	0.108	0.295	0.260	0.985
Q04	0.939	0.359	0.509	-2.219	-0.228	0.008	0.621
Q05	0.511	0.692	0.561	-0.104	-2.814	0.395	0.749
Q06	0.532	0.734	0.460	-0.028	0.392	-3.393	1.295

Inverse Form Retrieved from Direct Model

	Q01	Q02	Q03	Q04	Q05	Q06	Scale
P01	-0.460	-0.230	-0.133	-0.037	-0.069	-0.066	-0.995
P02	-0.215	-0.438	-0.146	-0.025	-0.073	-0.069	-0.965
P03	-0.187	-0.220	-0.405	-0.031	-0.080	-0.070	-0.993
P04	-0.254	-0.198	-0.158	-0.477	-0.017	-0.047	-1.151
P05	-0.187	-0.212	-0.152	-0.003	-0.413	-0.089	-1.056
P06	-0.163	-0.184	-0.124	-0.012	-0.085	-0.339	-0.907

Mixed Form Retrieved from Direct Model

	P01	P02	P03	P04	Q05	Q06	Expenditure
Q01	-3.058	1.301	0.484	0.137	-0.075	-0.073	1.124
Q02	1.224	-3.329	0.785	0.026	-0.096	-0.091	1.285
Q03	0.682	1.181	-3.157	0.094	-0.117	-0.090	1.189
Q04	0.894	0.297	0.460	-2.210	0.082	0.007	0.550
P05	0.207	0.281	0.222	-0.039	-0.361	-0.042	0.325
P06	0.181	0.249	0.161	-0.013	-0.042	-0.300	0.419

Direct Form Retrieved from Inverse Model

	P01	P02	P03	P04	P05	P06	Expenditure
Q01	-3.841	1.327	0.731	0.173	0.502	0.135	0.972
Q02	1.261	-4.139	1.086	0.144	0.477	0.106	1.065
Q03	1.029	1.604	-4.684	0.159	0.849	0.061	0.981
Q04	1.080	0.967	0.709	-3.244	-0.174	0.010	0.652
Q05	1.292	1.292	1.523	-0.081	-5.132	0.322	0.785
Q06	0.274	0.224	0.048	-0.029	0.304	-2.185	1.363

Inverse Form Estimated from Inverse Model

	Q01	Q02	Q03	Q04	Q05	Q06	Scale
P01	-0.427	-0.239	-0.159	-0.038	-0.092	-0.056	-1.014
P02	-0.227	-0.420	-0.168	-0.036	-0.091	-0.053	-0.997
P03	-0.224	-0.250	-0.347	-0.037	-0.104	-0.051	-1.015
P04	-0.247	-0.247	-0.169	-0.339	-0.066	-0.043	-1.112
P05	-0.234	-0.243	-0.187	-0.025	-0.275	-0.072	-1.038
P06	-0.111	-0.109	-0.069	-0.008	-0.061	-0.481	-0.840

Mixed Form Retrieved from Inverse Model

	P01	P02	P03	P04	Q05	Q06	Expenditure
Q01	-3.687	1.476	0.891	0.163	-0.102	-0.077	1.157
Q02	1.403	-4.000	1.236	0.134	-0.097	-0.063	1.226
Q03	1.262	1.834	-4.424	0.144	-0.169	-0.053	1.185
Q04	1.036	0.923	0.657	-3.241	0.034	0.001	0.625
P05	0.262	0.260	0.301	-0.017	-0.197	-0.029	0.194
P06	0.162	0.139	0.064	-0.016	-0.027	-0.462	0.651

Direct Form Retrieved from Mixed Model

	P01	P02	P03	P04	P05	P06	Expenditure
Q01	-4.619	2.002	1.146	0.148	0.215	0.223	1.055
Q02	1.993	-5.092	0.946	0.119	0.837	0.442	1.147
Q03	1.820	1.533	-5.867	0.032	1.191	1.036	0.875
Q04	1.330	1.148	0.325	-5.958	2.108	1.224	0.164
Q05	0.022	1.632	1.688	0.778	-5.494	-1.123	0.847
Q06	0.098	0.763	1.662	0.468	-1.332	-4.314	1.277

Inverse Form Retrieved from Mixed Model

	Q01	Q02	Q03	Q04	Q05	Q06	Scale
P01	-0.405	-0.256	-0.168	-0.033	-0.086	-0.075	-1.022
P02	-0.246	-0.393	-0.163	-0.034	-0.100	-0.076	-1.012
P03	-0.245	-0.248	-0.318	-0.034	-0.106	-0.097	-1.048
P04	-0.232	-0.245	-0.164	-0.205	-0.137	-0.099	-1.082
P05	-0.157	-0.201	-0.144	-0.044	-0.265	-0.007	-0.818
P06	-0.124	-0.136	-0.129	-0.029	0.007	-0.293	-0.703

Mixed Form Estimated from Mixed Model

	P01	P02	P03	P04	Q05	Q06	Expenditure
Q01	-4.614	2.083	1.267	0.191	-0.028	-0.044	1.136
Q02	2.002	-4.819	1.287	0.257	-0.136	-0.067	1.348
Q03	1.843	1.959	-5.256	0.255	-0.169	-0.196	1.269
Q04	1.357	1.846	1.218	-5.604	-0.336	-0.196	0.699
P05	-0.001	0.278	0.244	0.127	-0.194	0.051	0.100
P06	0.023	0.091	0.310	0.069	0.060	-0.247	0.265

Comparison of Own/Expenditure Elasticities in Ordinary Form

Own	Direct	Inverse	Mixed	Direct	Inverse	Mixed	Expenditure
Q01	-3.135	-3.841	-4.619	0.973	0.972	1.055	Coke, Sprite
Q02	-3.462	-4.139	-5.092	1.095	1.065	1.147	Pepsi, Mt. Dew
Q03	-3.264	-4.684	-5.867	0.985	0.981	0.875	7-up, Dr Pepper
Q04	-2.219	-3.244	-5.958	0.621	0.652	0.164	Lipton Brisk
Q05	-2.814	-5.132	-5.494	0.749	0.785	0.847	A&W, Rite Cola
Q06	-3.393	-2.185	-4.314	1.295	1.363	1.277	Sunkist,Canada

Comparison of Own/Scale Flexibilities in Inverse Form

Own	Direct	Inverse	Mixed	Direct	Inverse	Mixed	Sclae
P01	-0.460	-0.427	-0.405	-0.995	-1.014	-1.022	Coke, Sprite
P02	-0.438	-0.420	-0.393	-0.965	-0.997	-1.012	Pepsi, Mt. Dew
P03	-0.405	-0.347	-0.318	-0.993	-1.015	-1.048	7-up, Dr Pepper
P04	-0.477	-0.339	-0.205	-1.151	-1.112	-1.082	Lipton Brisk
P05	-0.413	-0.275	-0.265	-1.056	-1.038	-0.818	A&W, Rite Cola
P06	-0.339	-0.481	-0.293	-0.907	-0.840	-0.703	Sunkist,Canada

Comparison of Own/Expenditure Elasticities in Mixed Form

Own	Direct	Inverse	Mixed	Direct	Inverse	Mixed	Expenditure
Q01	-3.058	-3.687	-4.614	1.124	1.157	1.136	Coke, Sprite
Q02	-3.329	-4.000	-4.819	1.285	1.226	1.348	Pepsi, Mt. Dew
Q03	-3.157	-4.424	-5.256	1.189	1.185	1.269	7-up, Dr Pepper
Q04	-2.210	-3.241	-5.604	0.550	0.625	0.699	Lipton Brisk
P05	-0.361	-0.197	-0.194	0.325	0.194	0.100	A&W, Rite Cola
P06	-0.300	-0.462	-0.247	0.419	0.651	0.265	Sunkist,Canada

* P and Q denotes representative price and quantity indices for each group defended as Group01: Coca-Cola and Sprite, Group02: Pepsi-Cola and Mountain Dew, Group03: Seven-Up and Dr Pepper, Group04: Lipton Brisk., Group05: A&W and Rite-Cola, Group06: Sunkist and Canada Dry.

The magnitudes of consumers' response measured in three different specifications are different in general and some differences are not trivial. For the group 05 (A&W and Rite Cola) as an example, (a) The percentage increase in consumption with respect to one percent increase in total expenditure measured in the direct, inverse, and mixed demand systems are 0.749, 0.785, and 0.847 represented in the direct demand form. (b) The percentage decrease in normalized price with respect to one percent increase in the proportionate increase in each consumption measured in the direct, inverse, and mixed demand systems are 1.056, 1.038, and 0.818 represented in the inverse demand form. (c) The percentage increase in willingness to pay with respect to one percent increase in total expenditure measured in the direct, inverse, and mixed demand systems are 0.325, 0.194, and 0.100 represented in the mixed demand form. (d) The percentage decrease in consumption with respect to one percent increase in its own price measured in the direct, inverse, and mixed demand systems are 2.814, 5.132, and 5.494 represented in the direct demand form. (e) The percentage decrease in normalized price with respect to one percent increase in its own consumption measured in the direct, inverse, and mixed demand systems are 0.413, 0.275, and 0.265 represented in the inverse demand form. (f) The percentage decrease in willingness to pay with respect to one percent increase in its own consumption measured in the direct, inverse, and mixed demand systems are 0.361, 0.197, and 0.194 represented in the mixed demand form. Recall that these relationships are based on the partitioning quantity-dependent and price-dependent groups of commodities or the legitimate mixed demand system, which is identified by the PC algorithm.

Given the observation that the magnitudes of consumers' response measured in three different specifications are different in general, interpretations of the overall empirical results are not easy. One plausible comparison among three different demand systems of the direct, inverse, and mixed demand systems is possible based on the model selection approach. Given that all three competing models have the same number of independent parameters (23), all three model selection rules, the Akaike Information, Schwarz information criterion, and the Pollak and Wales' likelihood dominance criterion, are used based on the comparison of the estimated log-likelihood function values, such as the higher log-likelihood value, the higher ranking among competing models. The estimated log-likelihood values are 2698.77, 1332.23, 1269.15 for the inverse, direct, and mixed demand system, respectively. This result suggests that the inverse demand specification strongly dominates both the direct and the mixed demand specifications and the direct demand specification statistically dominates the mixed demand specifications. Note that this ordering of the statistical dominance is interpreted as the ranking among the competing models rather than the rejection one of the competing models. Additional empirical result that might lead one to prefer the inverse demand system is that the overall standard errors for the flexibility estimates of the inverse demand system are smaller than the overall standard errors for the elasticity estimates of the direct and mixed demand system. For example, the simple average of standard errors for the inverse, direct, and mixed uncompensated flexibility/elasticity estimates are 0.009, 0.159, and 0.164 respectively. Given that the inverse demand system identified through the application of the GES algorithm statistically dominate the other two specifications, it can be argued that the graphical causal model, especially the GES algorithm, provides reliable guidance for the choice among the direct, inverse, mixed demand systems.

On the other hand, it can be also argued that the information inferred by the PC algorithm is also useful, given the observations that (i) the comparisons among three different specifications are possible due to the reasonable partitioning of quantity-dependent and price-dependent groups of commodities or legitimate mixed demand system, which is identified by the PC algorithm. (ii) The magnitudes of consumers' response measured in three different specifications do not deviate too far with each other and thus provide plausible bounds in all the three different forms, although they are different in general and some differences are not trivial. In this respect, another possible approach to interpret the overall empirical results is to pursue the model averaging method rather than model selection method taken in this study, given that the model selection ordering of the statistical dominance need to be interpreted as the ranking among the competing models, rather than the rejection one of the competing models and accepting the other. Given that a whole family of mixed demand systems exists depending on the different partitioning of quantity-dependent and price-dependent groups of commodities, the overall results imply that the graphical causal methods of the PC and GES algorithms can provide reliable and informative guidance for the local identification issue of the choice among the direct, inverse, and mixed demand systems.

IV. Concluding Remarks

For the full use of the theoretical development to derive three alternative demand specifications of the direct, inverse, and mixed demand systems, the empirical procedure is proposed to address three issues of the identification, functional form, and comparisons among three specifications. The validity of the proposed procedure is illustrated by using retail checkout scanner data of soft drinks products. For the local (causal) identification issue between price and quantity variables among three possible specifications, the graphical causal models of the PC and GES algorithms are used. The GES algorithm result implies the inverse demand specification, whereas the PC algorithm result suggests the mixed demand system. Based on these inductively inferred local causal structures between price and quantity variables of a particular product, the inverse and mixed demand systems are estimated as well as the direct demand system for comparison purpose. To minimize the effect of different parameterization assumptions in the differential family of Rotterdam, LA/AIDS, NBR, and CBS, the generalized functional forms are derived for all the three demand systems. In all three demand systems, four nested parameterizations are statistically rejected and the synthetic differential functional forms are used for three demand systems. Based on the partitioning of the price-dependent variable group (the A&W and Rite-Cola and the Sunkist and Canada Dry product groups) and the quantity-dependent variable group (all other three groups) in the mixed demand system, which is identified by the PC algorithm, the estimated elasticities and flexibilities of three specifications are compared in the direct, inverse, and mixed demand system forms. Finally, the model selection approach, such as the Akaike Information, Schwarz information criterion, and the Pollak and Wales' likelihood dominance criterion, is adopted to statistically compare the competing three demand systems. Statistical evidences imply that the data support the inverse demand system, which is identified by the GES algorithm. Overall the empirical evidences suggest that the graphical causal models of the PC and GES algorithms provide helpful and reliable guidance for the full use of the theoretical development of three alternative demand specifications of the direct, inverse, and mixed demand systems.

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APPENDIX A

The main idea to identify the relationships among the Rotterdam, LA/AIDS, CBS, NBR functional forms and to derive their connections to the synthetic and generalized functional forms are based on the differential relationships of $dw_n = \{w_n d \ln q_n\} + w_n d \ln p_n - w_n d \ln y$, $dw_n = \{w_n d \ln \pi_n\} + w_n d \ln q_n$, and $d \ln y = d \ln \bar{y} + d \ln p_A$. These differential relationships are derived as follows: (i) $dw_n = w_n d \ln q_n + w_n d \ln p_n - w_n d \ln y$ or $dw_n = w_n d \ln q_n + w_n d \ln \pi_n$ is obtained by taking total differentiation of the identity $w_n = {p_n q_n}/{y}$, and use the log differential property, (ii) similarly, $d \ln y \equiv \sum_{n=1}^{N} w_n d \ln q_n + \sum_{n=1}^{N} w_n d \ln p_n \equiv d \ln Q + d \ln P$ is obtained from the identity of $y \equiv \sum_{n=1}^{N} p_n q_n$ and $d \ln y \equiv d \ln \bar{y} + d \ln P_A$ is derived from $y \equiv \sum_{i=1}^{m} p_i q_i + \sum_{k=m+1}^{N} p_k q_k$, where $d \ln \bar{y} = d \ln Q_A + d \ln Q_B + d \ln P_B$.

The derived relationships are used to identify the relationships between the Rotterdam with the LA/AIDS functional forms as follows: In the direct demand functions, based on the relationship of $dw_n = \langle w_n d \ln q_n \rangle + w_n d \ln p_n - w_n [d \ln Q + d \ln P]$, the Rotterdam form of $w_n d \ln q_n = [w_n \varepsilon_n] d \ln Q + \sum_{n'=1}^{N} [w_n \varepsilon_{n,n'}^c] d \ln p_{n'}$ can be written as the LA/AIDS form of $dw_n = [w_n \varepsilon_n - w_n] d \ln Q + \sum_{n'=1}^{N} [w_n \varepsilon_{n,n'}^c - w_n (w_{n'} - \delta_{n,n'})] d \ln p_{n'}$ or $dw_n = \beta_n d \ln Q + \sum_{n'=1}^{N} \gamma_{n,n'} d \ln p_{n'}$ by using parameterization of $\beta_n = [w_n \varepsilon_n - w_n]$ and $\gamma_{n,n'} = [w_n \varepsilon_{n,n'}^c - w_n (w_{n'} - \delta_{n,n'})]$ through $dw_n = \left\langle w_n \varepsilon_n d \ln Q + \sum_{n'=1}^{N} w_n \varepsilon_{n,n'}^c d \ln p_{n'} \right\rangle + w_n \left[\sum_{n'=1}^{N} \delta_{n,n'} d \ln p_{n'} \right] - w_n d \ln Q - w_n \left[\sum_{n'=1}^{N} w_{n'} d \ln p_{n'} \right]$. Whereas for the inverse demand functions, using $dw_n = \langle w_n d \ln \pi_n \rangle + w_n d \ln q_n + w_n d \ln Q - w_n d \ln Q$, the Rotterdam can be written as the LA/AIDS form of $dw_n = \beta_n d \ln Q + \sum_{n'=1}^{N} \gamma_{n,n'} d \ln q_{n'}$ by using parameterization of $\beta_n = [w_n f_n + w_n]$ and $\gamma_{n,n'} = [w_n f_{n,n'}^c - w_n (w_{n'} - \delta_{n,n'})]$ through $dw_n = \left\langle [w_n f_n] d \ln Q + \sum_{n'=1}^{N} w_n f_{n,n'}^c d \ln q_{n'} \right\rangle + w_n \left[\sum_{n'=1}^{N} \delta_{n,n'} d \ln q_{n'} \right] + w_n d \ln Q - w_n \left[\sum_{n'=1}^{N} w_{n'} d \ln q_{n'} \right]$.

The derivations of the synthetic and generalized functional forms for the direct and inverse demand systems are also based on the same relationships. For the direct demand specifications, the synthetic form can be derived as follows:

$$w_n d\ln q_n = [w_n \varepsilon_n] d\ln Q + \sum_{n'=1}^{N} [w_n \varepsilon^c_{n,n'}] d\ln p_{n'}$$

$$w_n d\ln q_n = [w_n \varepsilon_n - \theta_1^O w_n + \theta_1^O w_n] d\ln Q + \sum_{n'=1}^{N} [w_n \varepsilon^c_{n,n'} - \theta_2^O w_n (w_{n'} - \delta_{n,n'}) + \theta_2^O w_n (w_{n'} - \delta_{n,n'})] d\ln p_{n'} ,$$

or $w_n d\ln q_n = [C_n + \theta_1^O w_n] d\ln Q + \sum_{n'=1}^{N} [C_{n,n'} + \theta_2^O w_n (w_{n'} - \delta_{n,n'})] d\ln p_{n'}$

which can be transformed into the generalized form as follows:

$$\begin{aligned} dw_n &\equiv \langle w_n d\ln q_n \rangle + w_n d\ln p_n - w_n [d\ln Q + d\ln P] \\ &= \langle w_n d\ln q_n \rangle + w_n \left[\sum_{n'=1}^{N} \delta_{n,n'} d\ln p_{n'} \right] - w_n d\ln Q - w_n \left[\sum_{n'=1}^{N} w_{n'} d\ln p_{n'} \right] \\ &= \langle w_n d\ln q_n \rangle - w_n d\ln Q - w_n \left[\sum_{n'=1}^{N} (w_{n'} - \delta_{n,n'}) d\ln p_{n'} \right] \end{aligned}$$

$$dw_n = [w_n \varepsilon_n - \theta_1^O w_n - (1-\theta_1^O) w_n] d\ln Q + \sum_{n'=1}^{N} [w_n \varepsilon^c_{n,n'} - \theta_2^O w_n (w_{n'} - \delta_{n,n'}) - (1-\theta_2^O) w_n (w_{n'} - \delta_{n,n'})] d\ln p_{n'} .$$

or $dw_n = [C_n - (1-\theta_1^O) w_n] d\ln Q + \sum_{n'=1}^{N} [C_{n,n'} - (1-\theta_2^O) w_n (w_{n'} - \delta_{n,n'})] d\ln p_{n'}$

For the inverse demand specifications, the synthetic form can be derived as follows:

$$w_n d\ln \pi_n = [w_n f_n] d\ln Q + \sum_{n'=1}^{N} [w_n f^c_{n,n'}] d\ln q_{n'}$$

$$w_n d\ln \pi_n = [w_n f_n + \theta_1^I w_n - \theta_1^I w_n] d\ln Q + \sum_{n'=1}^{N} [w_n f^c_{n,n'} - \theta_2^I w_n (w_{n'} - \delta_{n,n'}) + \theta_2^I w_n (w_{n'} - \delta_{n,n'})] d\ln q_{n'} ,$$

or $w_n d\ln \pi_n = [D_n - \theta_1^I w_n] d\ln Q + \sum_{n'=1}^{N} [D_{n,n'} + \theta_2^I w_n (w_{n'} - \delta_{n,n'})] d\ln q_{n'}$

which can be transformed into the generalized form as follows:

$$\begin{aligned} dw_n &\equiv w_n d\ln p_n + w_n d\ln q_n - w_n d\ln y \\ &= w_n [d\ln p_n - d\ln y] + w_n d\ln q_n \\ &= \langle w_n d\ln \pi_n \rangle + w_n d\ln q_n + (w_n d\ln Q - w_n d\ln Q) \\ &= \langle w_n d\ln \pi_n \rangle + w_n \left[\sum_{n'=1}^{N} \delta_{n,n'} d\ln q_{n'} \right] + w_n d\ln Q - w_n \left[\sum_{n'=1}^{N} w_{n'} d\ln q_{n'} \right] \\ &= \langle w_n d\ln \pi_n \rangle + w_n d\ln Q - w_n \left[\sum_{n'=1}^{N} (w_{n'} - \delta_{n,n'}) d\ln q_{n'} \right] \end{aligned}$$

$$dw_n = \left[w_n f_n + \theta_1^I w_n + \left(1-\theta_1^I\right)w_n\right] d\ln Q + \sum_{n'=1}^{N}\left[w_n f_{n,n'}^c - \theta_2^I w_n\left(w_{n'} - \delta_{n,n'}\right) - \left(1-\theta_2^I\right)w_n\left(w_{n'} - \delta_{n,n'}\right)\right] d\ln q_{n'}$$

or $dw_n = \left[D_n + \left(1-\theta_1^I\right)w_n\right] d\ln Q + \sum_{n'=1}^{N}\left[D_{n,n'} - \left(1-\theta_2^I\right)w_n\left(w_{n'} - \delta_{n,n'}\right)\right] d\ln q_{n'}$

Based on the common logics, the analogous synthetic and generalized functional forms for the mixed demand system can be derived. For the mixed demand specifications for quantity-dependent group A, the synthetic form can be derived as follows:

$$\begin{aligned} w_i d\ln q_i &= \left[w_i \varepsilon_i\right] \cdot d\ln \bar{y} \\ &\quad + \sum_{j=1}^{m}\left[w_i \varepsilon_{i,j}^c - w_i \varepsilon_i \cdot \left(\sum_{k=m+1}^{N} w_k \cdot p_{k,j}^c \right)\right] \cdot d\ln p_j \\ &\quad + \sum_{s=m+1}^{N}\left[w_i q_{i,s}^c - w_i \varepsilon_i \cdot \left(\sum_{r=m+1}^{N} w_r f_{r,s}^c \right)\right] \cdot d\ln q_s \end{aligned}$$

$$\begin{aligned} w_i d\ln q_i &= \left[w_i \varepsilon_i - \theta_1^M w_i + \theta_1^M w_i \right] \cdot d\ln \bar{y} \\ &\quad + \sum_{j=1}^{m}\left[w_i \varepsilon_{i,j}^c - \theta_2^M w_i\left(w_j - \delta_{i,j}\right) + \theta_2^M w_i\left(w_j - \delta_{i,j}\right)\right] \cdot d\ln p_j \\ &\quad + \sum_{j=1}^{m}\left[-\left(w_i \varepsilon_i - \theta_1^M w_i + \theta_1^M w_i\right) \cdot \left(\sum_{r=m+1}^{N} w_r \cdot p_{r,j}^c + \theta_2^M w_r w_j - \theta_2^M w_r w_j \right)\right] \cdot d\ln p_j \\ &\quad + \sum_{s=m+1}^{N}\left[w_i q_{i,s}^c - \theta_2^M w_i w_s + \theta_2^M w_i w_s \right] \cdot d\ln q_s \\ &\quad + \sum_{s=m+1}^{N}\left[-\left(w_i \varepsilon_i - \theta_1^M w_i + \theta_1^M w_i\right) \cdot \left(\sum_{r=m+1}^{N} w_r f_{r,s}^c - \theta_2^M w_r\left(w_s - \delta_{r,s}\right) + \theta_2^M w_r\left(w_s - \delta_{r,s}\right)\right)\right] \cdot d\ln q_s \end{aligned}$$

$$\begin{aligned} w_i d\ln q_i &= \left[\alpha_i + \theta_1^M w_i \right] \cdot d\ln \bar{y} \\ &\quad + \sum_{j=1}^{m}\left[\alpha_{i,j} + \theta_2^M w_i\left(w_j - \delta_{i,j}\right) - \left(\alpha_i + \theta_1^M w_i\right) \cdot \left(\sum_{r=m+1}^{N} -\gamma_{r,j} - \theta_2^M w_r w_j \right)\right] \cdot d\ln p_j \\ &\quad + \sum_{s=m+1}^{N}\left[g_{i,s} + \theta_2^M w_i w_s - \left(\alpha_i + \theta_1^M w_i\right) \cdot \left(\sum_{r=m+1}^{N} \beta_{r,s} + \theta_2^M w_r\left(w_s - \delta_{r,s}\right)\right)\right] \cdot d\ln q_s \end{aligned}$$

which can be transformed into the generalized form as follows:

$$\begin{aligned} dw_i &\equiv w_i d\ln q_i + w_i d\ln p_i - w_i\left[d\ln y\right] \\ &= w_i d\ln q_i + w_i d\ln p_i - w_i\left[d\ln y - d\ln P_A + d\ln P_A\right] \\ &= w_i d\ln q_i + w_i d\ln p_i - w_i\left[d\ln \bar{y} + d\ln P_A\right] \\ &= \left\langle w_i d\ln q_i \right\rangle + w_i d\ln p_i - w_i d\ln \bar{y} - w_i d\ln P_A \\ &= \left\langle w_i d\ln q_i \right\rangle + w_i\left[\sum_{j=1}^{m} \delta_{i,j} d\ln p_j\right] - w_i d\ln \bar{y} - w_i\left[\sum_{j=1}^{m} w_j d\ln p_j\right] \\ &= \left\langle w_i d\ln q_i \right\rangle - w_i d\ln \bar{y} - \sum_{j=1}^{m} w_i\left(w_j - \delta_{i,j}\right) d\ln p_j \end{aligned}$$

$$
\begin{aligned}
dw_i = & \left[\alpha_i + (\theta_1^M - 1) w_i \right] \cdot d \ln \bar{y} \\
& + \sum_{j=1}^{m} \left[\alpha_{i,j} + (\theta_2^M - 1) w_i (w_j - \delta_{i,j}) - (\alpha_i + \theta_1^M w_i) \cdot \left(\sum_{r=m+1}^{N} - \gamma_{r,j} - \theta_2^M w_r w_j \right) \right] \cdot d \ln p_j \\
& + \sum_{s=m+1}^{N} \left[g_{i,s} + \theta_2^M w_i w_s - (\alpha_i + \theta_1^M w_i) \cdot \left(\sum_{r=m+1}^{N} \beta_{r,s} + \theta_2^M w_r (w_s - \delta_{r,s}) \right) \right] \cdot d \ln q_s
\end{aligned}
$$

On the other hand, the synthetic form can be derived as follows for the mixed demand specifications for price-dependent group B:

$$
\begin{aligned}
w_k d \ln p_k = & \left[w_k f_k \right] \cdot d \ln \bar{y} \\
& + \sum_{j=1}^{m} \left[w_k p_{k,j}^c - w_k f_k \cdot \left(\sum_{r=m+1}^{N} w_r p_{r,j}^c \right) \right] \cdot d \ln p_j \\
& + \sum_{s=m+1}^{N} \left[w_k f_{k,s}^c - w_k f_k \cdot \left(\sum_{r=m+1}^{N} w_r f_{r,s}^c \right) \right] \cdot d \ln q_s
\end{aligned}
$$

$$
\begin{aligned}
w_k d \ln p_k = & \left[w_k f_k - \theta_1^M w_k + \theta_1^M w_k \right] \cdot d \ln \bar{y} \\
& + \sum_{j=1}^{m} \left[w_k p_{k,j}^c + \theta_2^M w_k w_j - \theta_2^M w_k w_j \right] \cdot d \ln p_j \\
& + \sum_{j=1}^{m} \left[- (w_k f_k - \theta_1^M w_k + \theta_1^M w_k) \cdot \left(\sum_{r=m+1}^{N} w_r p_{r,j}^c + \theta_2^M w_r w_j - \theta_2^M w_r w_j \right) \right] \cdot d \ln p_j \\
& + \sum_{s=m+1}^{N} \left[w_k f_{k,s}^c - \theta_2^M w_k (w_s - \delta_{k,s}) + \theta_2^M w_k (w_s - \delta_{k,s}) \right] \cdot d \ln q_s \\
& + \sum_{s=m+1}^{N} \left[- (w_k f_k - \theta_1^M w_k + \theta_1^M w_k) \cdot \left(\sum_{r=m+1}^{N} w_r f_{r,s}^c - \theta_2^M w_r (w_s - \delta_{r,s}) + \theta_2^M w_r (w_s - \delta_{r,s}) \right) \right] \cdot d \ln q_s
\end{aligned}
$$

$$
\begin{aligned}
w_k d \ln p_k = & \left[\beta_k + \theta_1^M w_k \right] \cdot d \ln \bar{y} \\
& + \sum_{j=1}^{m} \left[- \gamma_{k,j} - \theta_2^M w_k w_j - (\beta_k + \theta_1^M w_k) \cdot \left(\sum_{r=m+1}^{N} - \gamma_{r,j} - \theta_2^M w_r w_j \right) \right] \cdot d \ln p_j \\
& + \sum_{s=m+1}^{N} \left[\beta_{k,s} + \theta_2^M w_k (w_s - \delta_{k,s}) - (\beta_k + \theta_1^M w_k) \cdot \left(\sum_{r=m+1}^{N} \beta_{r,s} + \theta_2^M w_r (w_s - \delta_{r,s}) \right) \right] \cdot d \ln q_s
\end{aligned}
$$

which can be transformed into the generalized form as follows:

$$
\begin{aligned}
dw_k \equiv & \; w_k d \ln p_k + w_k d \ln q_k - w_k \left[d \ln y \right] \\
= & \; w_k d \ln p_k + w_k d \ln q_k - w_k \left[d \ln y - d \ln P_A + d \ln P_A \right] \\
= & \; w_k d \ln p_k + w_k d \ln q_k - w_k \left[d \ln \bar{y} + d \ln P_A \right] \\
= & \; \langle w_k d \ln p_k \rangle + w_k d \ln q_k - w_k d \ln \bar{y} - w_k d \ln P_A \\
= & \; \langle w_k d \ln p_k \rangle + w_k \left[\sum_{s=1}^{m} \delta_{k,s} d \ln q_s \right] - w_k d \ln \bar{y} - w_k \left[\sum_{j=1}^{m} w_j d \ln p_j \right] \\
= & \; \langle w_k d \ln p_k \rangle - w_k d \ln \bar{y} - \sum_{j=1}^{m} w_k w_j d \ln p_j + \sum_{s=1}^{m} w_k \delta_{k,s} d \ln q_s
\end{aligned}
$$

$$
\begin{aligned}
dw_k = & \left[\beta_k + (\theta_1^M - 1) w_k \right] \cdot d \ln \bar{y} \\
& + \sum_{j=1}^{m} \left[- \gamma_{k,j} - (\theta_2^M + 1) w_k w_j - (\beta_k + \theta_1^M w_k) \cdot \left(\sum_{r=m+1}^{N} - \gamma_{r,j} - \theta_2^M w_r w_j \right) \right] \cdot d \ln p_j \\
& + \sum_{s=m+1}^{N} \left[\beta_{k,s} + \theta_2^M w_k w_s + (1 - \theta_2^M) w_k \delta_{k,s} - (\beta_k + \theta_1^M w_k) \cdot \left(\sum_{r=m+1}^{N} \beta_{r,s} + \theta_2^M w_r (w_s - \delta_{r,s}) \right) \right] \cdot d \ln q_s
\end{aligned}
$$

APPENDIX B

Direct demand system is related to mixed demand system by using following identities: $q_A^O\left[p_A, p_B^M(p_A, q_B, y), y\right] \equiv q_A^M(p_A, q_B, y)$ and $q_B^O\left[p_A, p_B^M(p_A, q_B, y), y\right] \equiv \overline{q_B^M}$. From identity of $q_A^O\left[p_A, p_B^M(p_A, q_B, y), y\right] \equiv q_A^M(p_A, q_B, y)$, (a) by differentiating identity w.r.t. ∇q_B, we get $\frac{\nabla q_A^O}{\nabla p_B}\frac{\nabla p_B^M}{\nabla q_B} = \frac{\nabla q_A^M}{\nabla q_B}$ or $\frac{\nabla q_A^O}{\nabla p_B} = \frac{\nabla q_A^M}{\nabla q_B}\left(\frac{\nabla p_B^M}{\nabla q_B}\right)^{-1}$, which can be written as $\left(\frac{\nabla q_A^O}{\nabla p_B}\frac{p_B}{q_A}\right) = \left(\frac{\nabla q_A^M}{\nabla q_B}\frac{q_B}{q_A}\right)\left(\frac{\nabla p_B^M}{\nabla q_B}\frac{q_B}{p_B}\right)^{-1}$ or $E_{AB}^O = Q_{AB}^M \cdot \left(F_{BB}^M\right)^{-1}$, (b) by differentiating w.r.t. ∇p_B, we get $\frac{\nabla q_A^O}{\nabla p_A} + \frac{\nabla q_A^O}{\nabla p_B}\frac{\nabla p_B^M}{\nabla p_A} = \frac{\nabla q_A^M}{\nabla p_A}$ or $\frac{\nabla q_A^O}{\nabla p_A} = \frac{\nabla q_A^M}{\nabla p_A} - \frac{\nabla q_A^O}{\nabla p_B}\frac{\nabla p_B^M}{\nabla p_A}$ which, using $\frac{\nabla q_A^O}{\nabla p_B} = \frac{\nabla q_A^M}{\nabla q_B}\left(\frac{\nabla p_B^M}{\nabla q_B}\right)^{-1}$, can be written as $\frac{\nabla q_A^O}{\nabla p_A} = \frac{\nabla q_A^M}{\nabla p_A} - \left[\frac{\nabla q_A^M}{\nabla q_B}\left(\frac{\nabla p_B^M}{\nabla q_B}\right)^{-1}\right]\frac{\nabla p_B^M}{\nabla p_A}$ or $E_{AA}^O = E_{AA}^M - Q_{AB}^M \cdot \left(F_{BB}^M\right)^{-1} P_{BA}^M$ through the relation of $\left(\frac{\nabla q_A^O}{\nabla p_A}\frac{p_A}{q_A}\right) = \left(\frac{\nabla q_A^M}{\nabla p_A}\frac{p_A}{q_A}\right) - \left(\frac{\nabla q_A^M}{\nabla q_B}\frac{q_B}{q_A}\right)\left(\frac{\nabla p_B^M}{\nabla q_B}\frac{q_B}{p_B}\right)^{-1}\left(\frac{\nabla p_B^M}{\nabla p_A}\frac{p_A}{p_B}\right)$, and (c) by differentiating w.r.t. ∇y, we also get $\frac{\nabla q_A^O}{\nabla p_B}\frac{\nabla p_B^M}{\nabla y} + \frac{\nabla q_A^O}{\nabla y} = \frac{\nabla q_A^M}{\nabla y}$ or $\frac{\nabla q_A^O}{\nabla y} = \frac{\nabla q_A^M}{\nabla y} - \frac{\nabla q_A^O}{\nabla p_B}\frac{\nabla p_B^M}{\nabla y}$, which, using $\frac{\nabla q_A^O}{\nabla p_B} = \frac{\nabla q_A^M}{\nabla q_B}\left(\frac{\nabla p_B^M}{\nabla q_B}\right)^{-1}$ again, can be written as $\frac{\nabla q_A^O}{\nabla y} = \frac{\nabla q_A^M}{\nabla y} - \left[\frac{\nabla q_A^M}{\nabla q_B}\left(\frac{\nabla p_B^M}{\nabla q_B}\right)^{-1}\right]\frac{\nabla p_B^M}{\nabla y}$ or $E_A^O = E_A^M - Q_{AB}^M \cdot \left(F_{BB}^M\right)^{-1} F_B^M$ through $\left(\frac{\nabla q_A^O}{\nabla y}\frac{y}{q_A}\right) = \left(\frac{\nabla q_A^M}{\nabla y}\frac{y}{q_A}\right) - \left(\frac{\nabla q_A^M}{\nabla q_B}\frac{q_B}{q_A}\right)\left(\frac{\nabla p_B^M}{\nabla q_B}\frac{q_B}{p_B}\right)^{-1}\left(\frac{\nabla p_B^M}{\nabla y}\frac{y}{p_B}\right)$.

From identities of $q_B^O\left[p_A, p_B^M(p_A, q_B, y), y\right] \equiv \overline{q_B^M}$, (a) by differentiating w.r.t. ∇q_B, we get $\frac{\nabla q_B^O}{\nabla p_B}\frac{\nabla p_B^M}{\nabla q_B} = 1$ or $\frac{\nabla q_B^O}{\nabla p_B} = \left(\frac{\nabla p_B^M}{\nabla q_B}\right)^{-1}$, which equal to $\left(\frac{\nabla q_B^O}{\nabla p_B}\frac{p_B}{q_B}\right) = \left(\frac{\nabla p_B^M}{\nabla q_B}\frac{q_B}{p_B}\right)^{-1}$ or $E_{BB}^O = \left(F_{BB}^M\right)^{-1}$, (b) by differentiating w.r.t. ∇p_A, we get $\frac{\nabla q_B^O}{\nabla p_A} + \frac{\nabla q_B^O}{\nabla p_B}\frac{\nabla p_B^M}{\nabla p_A} = 0$ or $\frac{\nabla q_B^O}{\nabla p_A} = -\frac{\nabla q_B^O}{\nabla p_B}\frac{\nabla p_B^M}{\nabla p_A}$, which, using

$\frac{\nabla q_B^O}{\nabla p_B} = \left(\frac{\nabla p_B^M}{\nabla q_B}\right)^{-1}$, can be written as $\frac{\nabla q_B^O}{\nabla p_A} = -\left(\frac{\nabla p_B^M}{\nabla q_B}\right)^{-1} \frac{\nabla p_B^M}{\nabla p_A}$ or $E_{BA}^O = -(F_{BB}^M)^{-1} P_{BA}^M$ through $\left(\frac{\nabla q_B^O}{\nabla p_A}\frac{p_A}{q_B}\right) = -\left(\frac{\nabla p_B^M}{\nabla q_B}\frac{q_B}{p_B}\right)^{-1}\left(\frac{\nabla p_B^M}{\nabla p_A}\frac{p_A}{p_B}\right)$, and (c) by differentiating w.r.t. ∇y, we get $\frac{\nabla q_B^O}{\nabla p_B}\frac{\nabla p_B^M}{\nabla y} + \frac{\nabla q_B^O}{\nabla y} = 0$ or $\frac{\nabla q_B^O}{\nabla y} = -\frac{\nabla q_B^O}{\nabla p_B}\frac{\nabla p_B^M}{\nabla y}$, which, using $\frac{\nabla q_B^O}{\nabla p_B} = \left(\frac{\nabla p_B^M}{\nabla q_B}\right)^{-1}$ again, can be written as $\frac{\nabla q_B^O}{\nabla y} = -\left(\frac{\nabla p_B^M}{\nabla q_B}\right)^{-1}\frac{\nabla p_B^M}{\nabla y}$ or $E_B^O = -(F_{BB}^M)^{-1} F_B^M$ through the relation of $\left(\frac{\nabla q_B^O}{\nabla y}\frac{y}{q_B}\right) = -\left(\frac{\nabla p_B^M}{\nabla q_B}\frac{q_B}{p_B}\right)^{-1}\left(\frac{\nabla p_B^M}{\nabla y}\frac{y}{p_B}\right)$.

Inverse demand system is related to mixed demand system by using following identities: $p_A^I[q_A^M(p_A, q_B, y), q_B, y] \equiv \overline{p_A}$ and $p_B^I[q_A^M(p_A, q_B, y), q_B, y] \equiv p_B^M(p_A, q_B, y)$ which are implied by $\pi_A^I[q_A^M(\pi_A, q_B, 1), q_B, 1] \equiv \overline{\pi_A}$ and $\pi_B^I[q_A^M(\pi_A, q_B, 1), q_B, 1] \equiv \pi_B^M(\pi_A, q_B, 1)$ through the relationships of $\pi_A^I[q_A^M(\pi_A, q_B, 1), q_B, 1]\cdot y \equiv \overline{\pi_A}\cdot y$ and $\pi_B^I[q_A^M(\pi_A, q_B, 1), q_B, 1]\cdot y \equiv \pi_B^M(\pi_A, q_B, 1)\cdot y$. From identities of $p_A^I[q_A^M(p_A, q_B, y), q_B, y] \equiv \overline{p_A}$, (a) by differentiating w.r.t. ∇p_A, we get $\frac{\nabla p_A^I}{\nabla q_A}\frac{\nabla q_A^M}{\nabla p_A} = 1$ or $\frac{\nabla p_A^I}{\nabla q_A} = \left(\frac{\nabla q_A^M}{\nabla p_A}\right)^{-1}$, which equals to $\left(\frac{\nabla p_A^I}{\nabla q_A}\frac{q_A}{p_A}\right) \equiv \left(\frac{\nabla \pi_A^I}{\nabla q_A}\frac{q_A}{\pi_A}\right) = \left(\frac{\nabla q_A^M}{\nabla p_A}\frac{p_A}{q_A}\right)^{-1}$ or $F_{AA}^I = (E_{AA}^M)^{-1}$, (b) by differentiating w.r.t. ∇q_B, we get $\frac{\nabla p_A^I}{\nabla q_A}\frac{\nabla q_A^M}{\nabla q_B} + \frac{\nabla p_A^I}{\nabla q_B} = 0$ or $\frac{\nabla p_A^I}{\nabla q_B} = -\frac{\nabla p_A^I}{\nabla q_A}\frac{\nabla q_A^M}{\nabla q_B}$, which, using $\frac{\nabla p_A^I}{\nabla q_A} = \left(\frac{\nabla q_A^M}{\nabla p_A}\right)^{-1}$, can be written as $\frac{\nabla p_A^I}{\nabla q_B} = -\left(\frac{\nabla q_A^M}{\nabla p_A}\right)^{-1}\frac{\nabla q_A^M}{\nabla q_B}$ or $\left(\frac{\nabla p_A^I}{\nabla q_B}\frac{q_B}{p_A}\right) \equiv \left(\frac{\nabla \pi_A^I}{\nabla q_B}\frac{q_B}{\pi_A}\right) = -\left(\frac{\nabla q_A^M}{\nabla p_A}\frac{p_A}{q_A}\right)^{-1}\left(\frac{\nabla q_A^M}{\nabla q_B}\frac{q_B}{q_A}\right)$, which in turn equal to $F_{AB}^I = -(E_{AA}^M)^{-1} Q_{AB}^M$.

From identity of $p_B^I[q_A^M(p_A, q_B, y), q_B, y] \equiv p_B^M(p_A, q_B, y)$, (a) by differentiating identity w.r.t. ∇p_A, we get $\frac{\nabla p_B^I}{\nabla q_A}\frac{\nabla q_A^M}{\nabla p_A}=\frac{\nabla p_B^M}{\nabla p_A}$ or $\frac{\nabla p_B^I}{\nabla q_A}=\frac{\nabla p_B^M}{\nabla p_A}\left(\frac{\nabla q_A^M}{\nabla p_A}\right)^{-1}$ which can be written as $\left(\frac{\nabla p_B^I}{\nabla q_A}\frac{q_A}{p_B}\right)\equiv\left(\frac{\nabla \pi_B^I}{\nabla q_A}\frac{q_A}{\pi_B}\right)=\left(\frac{\nabla p_B^M}{\nabla p_A}\frac{p_A}{p_B}\right)\left(\frac{\nabla q_A^M}{\nabla p_A}\frac{p_A}{q_A}\right)^{-1}$ or $F_{BA}^I=P_{BA}^M\left(E_{AA}^M\right)^{-1}$, (b) by differentiating w.r.t. ∇q_B, we get $\frac{\nabla p_B^I}{\nabla q_A}\frac{\nabla q_A^M}{\nabla q_B}+\frac{\nabla p_B^I}{\nabla q_B}=\frac{\nabla p_B^M}{\nabla q_B}$ or $\frac{\nabla p_B^I}{\nabla q_B}=\frac{\nabla p_B^M}{\nabla q_B}-\frac{\nabla p_B^I}{\nabla q_A}\frac{\nabla q_A^M}{\nabla q_B}$ which, using $\frac{\nabla p_B^I}{\nabla q_A}=\frac{\nabla p_B^M}{\nabla p_A}\left(\frac{\nabla q_A^M}{\nabla p_A}\right)^{-1}$, can be written as $\frac{\nabla p_B^I}{\nabla q_B}=\frac{\nabla p_B^M}{\nabla q_B}-\left[\frac{\nabla p_B^M}{\nabla p_A}\left(\frac{\nabla q_A^M}{\nabla p_A}\right)^{-1}\right]\frac{\nabla q_A^M}{\nabla q_B}$ or $\left(\frac{\nabla p_B^I}{\nabla q_B}\frac{q_B}{p_B}\right)\equiv\left(\frac{\nabla \pi_B^I}{\nabla q_B}\frac{q_B}{\pi_B}\right)=\left(\frac{\nabla p_B^M}{\nabla q_B}\frac{q_B}{p_B}\right)-\left(\frac{\nabla p_B^M}{\nabla p_A}\frac{p_A}{p_B}\right)\left(\frac{\nabla q_A^M}{\nabla p_A}\frac{p_A}{q_A}\right)^{-1}\left(\frac{\nabla q_A^M}{\nabla q_B}\frac{q_B}{q_A}\right)$, which in turn equal to $F_{BB}^I=F_{BB}^M-P_{BA}^M\left(E_{AA}^M\right)^{-1}Q_{AB}^M$. From the relation $f_n=\sum_{n'=1}^N f_{n,n'}$ or $F_N^I=RowSum\left(F_{N,N}^I\right)$ of inverse demand function, we get $F_A^I=RowSum\left(F_{AA}^I \vdots F_{AB}^I\right)$ and $F_B^I=RowSum\left(F_{BA}^I \vdots F_{BB}^I\right)$. Using $F_{AA}^I=\left(E_{AA}^M\right)^{-1}$ and $F_{AB}^I=-\left(E_{AA}^M\right)^{-1}Q_{AB}^M$, we can write $F_A^I=RowSum\left[\left(E_{AA}^M\right)^{-1}\vdots-\left(E_{AA}^M\right)^{-1}Q_{AB}^M\right]$. Using $F_{BA}^I=P_{BA}^M\left(E_{AA}^M\right)^{-1}$ and $F_{BB}^I=F_{BB}^M-P_{BA}^M\left(E_{AA}^M\right)^{-1}Q_{AB}^M$, we can write $F_B^I=RowSum\left[P_{BA}^M\left(E_{AA}^M\right)^{-1}\vdots F_{BB}^M-P_{BA}^M\left(E_{AA}^M\right)^{-1}Q_{AB}^M\right]$.

From the resulting two kinds of relationships between the mixed and the direct and between the mixed and the inverse demand systems, the other implied relationships can also be derived between the direct and the inverse demand systems through their relationships with the mixed demand systems.