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Some Properties of Simple Functions of Random Variables

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Simple functions of random variables, such as sums, products, quotients and powers, arise in many applications in stochastic economics. It is sometimes convenient to determine the moments of such functions. Cases where this has been or can be done analytically are reviewed, and comments offered on procedures for when analysis fails.

1 Introduction

Probabilistic microeconomics is a field of rapidly growing importance [26]. Economists of both theoretical and applied bents are increasingly giving explicit recognition to the impact of risk in economic phenomena. Agricultural economists have long been concerned with instability in markets and with the effects that market and climatic uncertainties have on farm families and on resource use [2, 9, 15, 25, 26, 34].

Attempts at modelling risky phenomena inevitably lead to descriptions couched in terms of probability distributions (e.g., of prices and outputs). Analysts thus are increasingly confronted with the problems of dealing with mathematical functions of one or more random variables. It is here that practitioners may encounter difficulties on their selected analytical path.

Dealing with functions of random variables has long been the province of mathematical statistics, and some kindred disciplines. Knowledge in this area is probably not readily accessible to, or appreciated by, many potential practitioners in, for example, agricultural economics. For instance, a survey of major instructional texts in economic statistics, mathematical statistics, mathematical economics and decision analysis reveals that this general topic is given virtually no attention whatsoever. (The rare exceptions concern sums of random variables, usually in connection with the Central Limit Theorem [27].) The limited material that is available is generally in either advanced statistics references or relatively obscure articles.

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Our intention is thus to point to what is presently available, to attempt to catalogue what can and cannot be done, and to detail the practicability of existing procedures.

2 The Simple Functions Considered

Four elementary functions are considered—namely, sums, products, quotients and powers of random variables. Let X and Y be random variables with specified distributions (constants are denoted by lower case symbols), then the problem under discussion is the nature of the derived random variables denoted by Z in:

(1) $Z = bX + cY,$

(2) $Z = XY,$

(3) $Z = X/Y,$

and

(4) $Z = X^d.$

More complex functions can then be built up from these elementary functions. For example, in stochastic budgeting, gross margin Z may be defined in terms of product price X, process yield Y, costs (that vary with yield) U, and other variable costs V in the equation

(5) $Z = (X - U)Y - V,$

which is a combination of the elementary equations (1) and (2).

3 A Guide to the Literature

Some results are available for some aspects of the distribution of Z and for some particular cases of distributions for X (and Y). These can be conveniently overviewed by means of the tabular classification of Table 1. The aspects of Z considered in Table 1 are the mean $E(Z)$, variance $V(Z) = \mu_Z(2)$, standardized coefficient of skewness $\alpha_3(Z)$ based on the third central moment $\mu_Z(3)$ and defined as $\alpha_3(Z) = \mu_Z(3)/V(Z)^{1.5}$ and the standardized coefficient of kurtosis $\alpha_4(Z)$ based on the fourth central moment $\mu_Z(4)$, $\alpha_4(Z) = \mu_Z(4)/V(Z)^2$, and $D(Z)$ the distribution of Z.

This overview reveals the severely restricted scope of available results. However, some general techniques are available which in principle permit the ready extension of results at least for the moments of simple functions of independent variables other than those mentioned in Table 1. For instance, practitioners may well wish to deal with functions of more-or-less common and convenient distributions such as the triangular, beta, etc. These techniques are reviewed briefly in the next section and are illustrated in the section that follows it.

Table 1: Overview of Published Results

Function	Aspect[g] of Z	Distribution of X and Y when from the same family					Different families
		Normal (Independent)	Normal (dependent)	Uniform (independent)	Gamma (independent)	Log-normal	
Sum ..	E(Z)	s[a]	s	s	s	s	s
	V(Z)	s	s	s	s	s	s
	$\alpha_3(Z)$	n[b]	n	s	s	0	0
	$\alpha_4(Z)$	n	n	s	s	0	0
	D(Z)	n	n	23[c]	s	0	ss[d]
Product ..	E(Z)	s	s	s	s	s	s
	V(Z)	5, 11, 12, 14	5, 11, 12	32	5, 11	n	5,11
	$\alpha_3(Z)$	11	11	0	0	n	0
	$\alpha_4(Z)$	11	11	0	0	n	0
	D(Z)	7	3	33	22	n	6
Quotient..	E(Z)	0	13	33	0	n	0
	V(Z)	0	13	33	0	n	0
	$\alpha_3(Z)$	0	0	0	0	n	0
	$\alpha_4(Z)$	0	0	0	0	n	0
	D(Z)	10, 16, 24	10, 16, 24	24, 32	21, 23	n	6
Power[e] ..	E(Z)	12	..	s	s[f]	n	..
	V(Z)	12	..	s	s[f]	n	..
	$\alpha_3(Z)$	12	..	s	s[f]	n	..
	$\alpha_4(Z)$	12	..	s	s[f]	n	..
	D(Z)	12	..	s	s[f]	n	..

a s denotes a standard result available in most elementary statistics books, or very easily derivable from standard results.

b n indicates that the results are available because the normal is a "stable" distribution. Hence linear combinations of normals are normal, products, quotients and powers of log-normal are log-normal.

c Other table entries refer to reference number, a zero if the relevant result is seemingly unavailable, and a dash if it is irrelevant.

d ss indicates that general formulae are available, but may be impossible to apply in specific cases.

e Column headings for this function are to be interpreted as, respectively "Distribution of X" and (for "Different Families") as "Families other than those special cases mentioned in the rest of the table".

These results hold only for the standard gamma distribution.

g See text for descriptions of functions.

4 A Useful Analytical Technique

It is convenient to introduce a mathematical device known as the Mellin transform [7, 31, 33, 35], in order to evaluate the moments of the elementary functions (2), (3) and (4). Suppose a random variable X (presumed non-negative) has a probability density function $f_X(x)$. The Mellin transform $M_X(s)$ of $f_X(x)$ is defined by the relation

$$(6) \qquad M_X(s) = E(X^{s-1}) = \int_0^\infty x^{s-1} f_X(x)\,dx.$$

Thus $M_X(s)$ evaluated at $s = 2, \ldots, 5$ gives the first, second, third and fourth moments of X about the origin. In general,

$$(7) \qquad M_X(n+1) = \mu'_X(n),$$

where $\mu'_X(n)$ is the n-th moment of X about the origin.

Consider now the product of $Z = XY$ of two independent random variables X and Y. It is known [30, p. 318] that the *PDF* $f_Z(z)$ of Z is given by

$$(8) \qquad f_Z(z) = \int_0^\infty f_X\left(\frac{z}{u}\right) f_Y(u) \frac{du}{u} = \int_0^\infty f_X(u) f_Y\left(\frac{z}{u}\right) \frac{du}{u}.$$

Here again there is a link with the Mellin transform. An important property (from the convolution theorem [7]) is that, if $f_Z(z)$ is defined as in (8), the corresponding transform $M_Z(s)$ is related to the transforms of $f_X(x)$ and $f_Y(y)$

$$(9) \qquad M_Z(s) = M_X(s)\, M_Y(s).$$

Thus, on setting $s = n + 1$, by (7) we have

$$(10) \qquad \mu'_Z(n) = \mu'_X(n)\, \mu'_Y(n),$$

or, the n-th moment of the product of X and Y is simply the product of the n-th moments of X and Y. This result could obviously be generalized to the product of any finite number of independent random variables.

Results for equations (4) and (3) can now be readily obtained, although progress on (1) requires a different tack.

Thus, for equation (4), that is $Z = X^d$,

$$(11) \qquad M_Z(s) = E(Z^{s-1}) = E(X^{ds-d}) = M_X(ds - d + 1),$$

by (6).

Again, on setting $s = n + 1$,

$$(12a) \qquad \mu'_Z(n) = M_X(nd + 1),$$

and if d is an integer,

$$(12b) \qquad \mu'_Z(n) = \mu'_X(nd).$$

For example, the mean of X^4 is simply the fourth moment (about the origin) of X.

For equation (3), $Z = X/Y = XY^{-1}$, we put $d = -1$ in (11) and use (9) to obtain

$$(13) \qquad M_Z(s) = M_X(s)\, M_Y(2 - s),$$

so that

$$(14) \qquad \mu'_Z(n) = \mu'_X(n)\, M_Y(1 - n),$$

when $M_Y(1 - n)$ exists—and it may well not exist, in which case the moments of the quotient will not exist.

The results (10) and (12b) could be obtained very simply by using well-known rules for expectations of independent random variables. The introduction of the Mellin transform serves two purposes. Firstly, the moments of not only products and integer powers but also of non-integer powers and quotients may be obtained ((12a) and (14)). Secondly, once the fundamental connection (7) between moments and Mellin transforms has been appreciated, use can be made of the very extensive tabulations of Mellin transforms [4, 29] to obtain difficult moments.

The moments of a linear combination of independent random variables $Z = aX + bY$ may be derived using the binomial theorem[1]

$$\mu'_Z(n) = E[aX + bY]^n$$

$$= \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r} E(X^r Y^{n-r})$$

$$= \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r} E(X^r)\, E(Y^{n-r})$$

as X and Y are independent.

Thus,

$$\mu'_Z(n) = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r} \mu'_X(r)\, \mu'_Y(n-r). \tag{15}$$

Using the usual definitions of the second, third and fourth *central* moments namely

$$\mu(2) = \mu'(2) - [\mu'(1)]^2, \tag{16a}$$

$$\mu(3) = \mu'(3) - 3\mu'(1)\,\mu'(2) + 2[\mu'(1)]^3, \tag{16b}$$

$$\mu(4) = \mu'(4) - 4\mu'(1)\,\mu'(3) + 6[\mu'(1)]^2\,\mu'(2) - 3[\mu'(1)]^4, \tag{16c}$$

the second, third and fourth central moments of Z are found from (15) to have the simple forms

$$\mu_Z(2) = a^2\,\mu_X(2) + b^2\,\mu_Y(2), \tag{17a}$$

$$\mu_Z(3) = a^3\,\mu_X(3) + b^3\,\mu_Y(3), \tag{17b}$$

and

$$\mu_Z(4) = a^4\,\mu_X(4) + b^4\,\mu_Y(4) + 6a^2b^2\,\mu_X(2)\,\mu_Y(2). \tag{17c}$$

5 When Analysis Fails

It has been demonstrated that analysis can proceed fairly successfully when component random variables are independently distributed, at least for most common distributions that are "flexible" and therefore likely to be useful. In other cases where statistical dependencies are involved, analysis presently fails for moments beyond the first two for other than "stable" distributions and there seems no operational approach alternative to the "brute force" methods based on Monte Carlo sampling.

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[1] We are indebted to a referee for pointing out to us that our earlier treatment of this topic using cumulants was unnecessarily complicated. However, our experience with calculating central moments is that use of cumulants yields simpler computations than the binomial theorem.

The Monte Carlo method in work of this type involves pseudorandom sampling from the respective distributions [8, 28], performing the arithmetic operations implied by the function under review, and processing the resulting composite variates in some informative way. Most simply, the processing might yield various summary statistics such as sample moments and order statistics. Alternatively, the variates might be used to fit plausible complete empirical or theoretical distributions.

Monte Carlo sampling is most straightforward when only independent (marginal) distributions are involved. The only case of multivariate (joint) distributions which is straightforward is the multivariate normal. Otherwise, dependencies must either be circumnavigated by using devices such as transformations of the variable analogous to those employed in avoiding subjective specification of multivariate probability distributions [2, Ch. 2] or handled by approximation procedures. Two types of approximation have appeal. An obvious one is to approximate a joint distribution as a multivariate normal, in which case dependence is captured by the pair-wise correlations. The validity of such an approximation could be examined by tests of normality on the marginal and some conditional distributions.

An alternative procedure [19] for pairs of jointly distributed variables is to take the marginal distribution of one as given, to specify the simple correlation between the two variables, to describe the second marginal distribution only by mean and variance, and thereby to specify an indeterminate joint distribution that is "correct" in terms of means, variances and covariances and in one marginal distribution. This alternative procedure is used in [1] for dealing with non-normal bivariate distributions.

6 An Illustration

Consider the case of two variables (whose moments are summarized in Table 2 below):

(i) X general beta, having *PDF*

$$f_X(x) = \frac{(x - a)^{c-1}(b - x)^{d-1}}{B(c, d)\,(b - a)^{c+d-1}}, \quad (a < x < b)$$

where $B(c, d)$ is the beta function. We will assign $a = 90$, $b = 150$, $c = 1$ and $d = 3$.

(ii) Y triangular, having *PDF*

$$f_Y(y) = \frac{2}{(b - a)}\frac{(y - a)}{(m - a)}, \qquad (a \leqslant y < m)$$

$$= \frac{2}{(b - a)}\frac{(b - y)}{(b - m)}, \qquad (m \leqslant y \leqslant b).$$

Here $a = 0.8$, $m = 1.2$, $b = 2.8$.

We will consider initially the product $Z = XY$. This could represent a definition of revenue Z in terms of price X (say in \$/t) and crop yield Y (say in t/ha). Suppose initially that X and Y are independent. What can be said about the nature of the distribution of Z? Intuitively, it will be of a non-standard, probably very complex type although the range is easily obtained.

Apart from that, however, the distribution can be usefully described by its first few moments.

Using the results of Section 5, and equation (10) in particular, the moments of Z can be found once the moments of X and Y are known, and these may be found by obtaining the Mellin transforms of the density functions of X and Y. The Mellin transform for the general beta distribution is [31]

(18) $$M(s) = b^{s-1} F(1 - s, d, c + d; (b - a)/b),$$

where F denotes the hypergeometric function [20]. Thus, by (7)

$$\mu'_X(n) = b^n F(-n, d, c + d; (b - a)/b).$$

Using a fundamental property of the hypergeometric function, namely that when the first parameter is a negative integer F may be expressed as a polynomial [20, p. 39], we obtain

(19a) $$\mu'_X(n) = b^n \left\{1 + \sum_{r=1}^{n} (-1)^r \binom{n}{r} \frac{(d)_r}{(c + d)^r} \left(\frac{b - a}{b}\right)^r\right\},$$

where the symbol $(\alpha)_r$ denotes $\alpha(\alpha + 1)(\alpha + 2) \ldots (\alpha + r - 1)$. This specializes to

(19b) $$\mu'_X(1) = (ad + bc)/(c + d),$$

(19c) $$\mu_X(2) = (b - a)^2 cd/[(c + d)^2(c + d + 1)],$$

(19d) $$\alpha_3(X) = [2(d - c)/(c + d + 2)]\,[1/c + 1/d + 1/cd]^{0\cdot5},$$

(19e) $$\alpha_4(X) = 3(c + d + 1)\,[2(c + d)^2 + cd(c + d - 6)]/[cd(c + d + 2)(c + d + 3)].$$

For the triangular distribution, the Mellin transform may be calculated directly as

(20) $$M(s) = 2\{(b - m)a^{s+1} + (m - a)b^{s+1} - (b - a)m^{s+1}\}/\{s(s + 1)(b - m)(m - a)(b - a)\},$$

and the moments about the origin are readily found (alternatively, see [17, p. 64]). Given the first four moments of X and Y, the corresponding moments of Z are obtained by equation (10). Central moments are found using (16a), (16b) and (16c), which in turn lead to the standardized descriptors α_3 and α_4. These results have been assembled in Table 2.

Illustrations of the other algebraic results can also be readily obtained. For instance, suppose Y had been measured as the square root of actual yield so that yield was now $Z = Y^2$. What, for example, is the variance of Z? This would be computed from the first two moments of Z which in turn are related to the moments of Y by equation (12b). Thus,

(21) $$\mu'_Z(1) = \mu'_Y((1)(2)) = \mu'_Y(2) = 2.74\dot{6},$$

and

(22) $$\mu'_Z(2) = \mu'_Y((2)(2)) = \mu'_Y(4) = 9.7502,$$

so that

(23) $$V(Z) = \mu_Z(2) = \mu'_Z(2) - [\mu'_Z(1)]^2 = 2.206,$$

and other moments could similarly be determined.

Table 2: Moments and Related Quantities for X, Y and Z[a]

Feature	X	Y	Z
$\mu'(1)$	105	1.6	168.0
$\mu'(2)$	11 160	$2.74\dot{6}$	30 652.8
$\mu'(3)$	1 201 500	5.0304	6 044 025.60
$\mu'(4)$	131 104 286	$9.75018\dot{6}$	1 278 291 261.34
$\mu(2)$	135	$0.18\dot{6}$	2 428.8
$\mu(3)$	1 350	0.0384	78 278.416
$\mu(4)$	56 410.719	$0.08362\dot{6}$	17 791 294
c^b	0.1107	0.2700	0.2934
α_3	0.861	0.4761	0.6540
α_4	3.095	2.4000	3.0159
σ	11.619	0.4320	49.283

a Where $Z = XY$, X is beta and Y triangular.
b The coefficient of variation.

Consider now the quotient $Z = X/Y$ where, for example, X might be revenue/ha and Y yield; so that Z now represents price. By equation (14) the moments about the origin of Z are given by

$$\mu'_Z(n) = \mu'_X(n) \quad M_Y(1 - n),$$

and $M_Y(s)$ by (20). Performing the necessary calculations for $n = 1$, 2, and 3, we obtain

$$M_Y(0) = 0.67184, \qquad M_Y(-1) = 0.48410, \qquad M_Y(-2) = 0.3720,$$

and these, when combined with $\mu'_X(1)$, $\mu'_X(2)$ and $\mu'_X(3)$ (as shown in Table 2) yield

$$\mu'_Z(1) = 70.5432, \qquad \mu'_Z(2) = 5402.556, \qquad \mu'_Z(3) = 446958.00$$

The coefficient of skewness $\alpha_3(Z)$, for example, is then easily found to be 0.6491. To complete this brief illustrative section, let us return to the initial example of the product $Z = XY$. Suppose now that price and yield are no longer judged to be independent, but are related in a manner crudely summarized by the simple correlation, $r = -0.5$. Unfortunately, the analytical procedures of Section 4 are now no longer applicable, except perhaps as pragmatic approximations.

The various approximation procedures discussed in Section 5 are accordingly applied to the example and the results summarized in Table 3 by means of the relatively digestible standardized descriptors based on the first four moments. Unfortunately, we learn little from this comparison except that different simplifying assumptions applied to an intractible problem lead, indeed, to different approximate solutions.

Table 3: Alternative Approaches to Estimating the Descriptors of $Z = XY$, when X and Y Are Dependent

Approach	Mean	c	α_3	α_4
1 Analysis assuming independence[a] ..	168.0	0.293	0.654	3.016
2 Monte Carlo assuming independence[b] ..	166.2	0.308	0.877	0.299
3 Monte Carlo with correlation and including only mean and variance of X.	163.7	0.233	0.244	−0.611
4 Monte Carlo with correlation but assuming X and Y normal.	174.1	0.286	0.262	0.087
5 Analysis with correlation but assuming X and Y normal[c].	165.5	0.241	−0.076	0.0199
6 Analysis assuming X and Y normal but ignoring correlation[c].	168.0	0.293	0.212	0.0196

[a] From Table 2. Symbols employed here are as defined in Table 2.

[b] Using [1] based on sample size of 200.

[c] Using Haldane's [12, p. 234] formulae.

7 Conclusion

Apart from some simple and very special cases, it is not easy to determine the complete nature of the probability distribution of even rather simple functions of two or more random variables.

However, when a distribution can be fairly adequately described in terms of its first few moments, the situation is not so bleak. In particular, if the variables are statistically independent, progress is relatively straightforward following procedures outlined in Section 4.

Non-independent variables pose more challenging problems for analysis. At this stage it seems best either to use the crude approach of Monte Carlo sampling to obtain approximate empirical solutions, or to use the analytical results based on the multivariate normal distribution, when this is appropriate.

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