An Examination of the Empirical Properties of Duality between the Restricted Profit, Unrestricted Profit, and Production Functions

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Abstract: This research examines the empirical properties of duality theory. A comparison of the Hessian matrices calculated from the normalized unrestricted and restricted profit, and production functions indicate that duality is highly sensitive to measurement error and relative price variability.

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Introduction

Historically, it was a great accomplishment to discover the dual relationship between functions. Duality theory has proven to be useful by providing alternative avenues for solving applied problems. According to Shumway (1995), between 1982 and 1995, more than 100 studies were published in agriculture economics journals that used neoclassical duality concepts. Prior to that time, less than a dozen articles had been published which used the same dual approach (Shumway, 1995). The use of dual relationships has given researchers flexibility in problem solving when data is limited or is of a specific type.

In theory, one is able to make appropriate conversions and migrate between the production function, the unrestricted profit function, the restricted profit function and/or the cost function. It is assumed that underlying every profit or cost function is a known technology or production function. Lau (1976) used Hessian identities to show that estimates from a profit function can be converted to a cost function and vice-versa under the assumption of perfect competition. Thus long-run effects may be obtained in two ways: either the unrestricted profit function can be estimated and long-run effects calculated directly, or the restricted profit or cost function may be estimated, and using Lau's (1976) matrix identities, long-run effects may be determined.

For the purposes of this paper, the primal problem will refer to a firm's profit maximization decision subject to a production function, and the dual approach will refer to the firm's short run profit maximizing decision given fixed commodities. Given the above definitions, advantages to using the dual approach include: 1) prices are exogenous to the decision maker, thus simultaneity may exist in determining input quantities, 2) measurement problems may exist with quantity data, 3) additional flexible forms can be estimated, thus less restrictions are placed on the technology. While the dual approach has some advantages, some disadvantages do exist. These disadvantages include: 1) prices may not be independent of output, such is the case under market power, 2) data is typically short-run and economists often wish to make long-run inferences, 4) adjustment costs, and 5) prices may be collinear. Shumway (1995) also pointed out several potential advantages and disadvantages to using the dual approach.

The quality of estimated dual relationships may not be good, especially if price variability is small, firms have market power, or if a large amount of noise is present in the data (measurement error). If empirically one can determine the reason why Lau's Hessian identities do not hold, then potential information may be gained. For example, if the dual relationship between the cost and profit function does not hold when firms have market power, and then differences in the estimation of elasticities using a restricted and unrestricted profit approach potentially could give an indication of the presence of market power. Additionally, one may be able determine that the dual relationship does not hold unless a certain level of price variability is achieved.

Motivation and Background

As a motivation for this study, we first estimated a normalized quadratic cost function and conditional demand functions with symmetry and homogeneity imposed on farm level data from the Kansas Farm Management Data Bank. To find the matrix of uncompensated elasticities, we used the Hessian identities from Lau (1976). When these uncompensated elasticities were compared with the results from an estimation of the unrestricted profit function, significant inconsistencies were found. What caused these inconsistencies in elasticities, which theoretically should have been identical? There may be several reasons for the difference in unconditional elasticities such as violation of regularity conditions, violation of perfect competition, inaccuracy of estimated functional form, large amounts of noise in the data, relatively small relative price variability, or measurement error in coefficient variables. The aim of this research is to examine reasons for these inconsistencies in the results.

Objectives

The objective of this research is to examine the empirical properties of duality theory. Specifically, we attempt to econometrically examine the Hessian identities develop by Lau (1976). No study has examined the ability to recover the original production technology. The effects of price variability and data noise will be examined as factors important to recovering the original production technology. Hessian matrices calculated from the unrestricted and restricted profit functions under different price variability and measurement error scenarios will be compared.

Methods and Procedures

To control all extraneous effects that may be involved in empirical estimation, data is simulated using Monte Carlo simulation techniques. The firm's profit maximization problem is used as a basis for the data generation process. A single-output, four input production function is assumed in the following quadratic form:

$$Y = a_1 x_1 + a_2 x_2 + a_3 x_3 + a_{41} x_4 + a_{11} x_1^2 + a_{22} x_2^2 + a_{33} x_3^2 + a_{44} x_4^2 + a_{12} x_1 x_2 + a_{13} x_1 x_3 +$$

$$a_{14} x_1 x_4 + a_{23} x_2 x_3 + a_{24} x_2 x_4 + a_{34} x_3 x_4$$

where Y is the output quantity and and the x's represent the input quantities. The intercept is set to zero, such that no output will be produced without any inputs. To maintain consistency with economic theory, the coefficient for the linear and off-diagonal quadratic terms are assumed to be positive, while the diagonal quadratic terms are assumed to be negative. This restricts the production function to be concave. Once the production function exits, the firm's maximization decision is as follows:

$$Max \quad p = P \cdot Y - \sum_{i=1}^{4} w_i x_i$$

where Y is the production function previously defined, P is the output price, and the w's represent input prices. Given prices, the firm "chooses" the amount of each input to use that maximizes profits. Once the total amount of each input use is determined, the output quantity, Y, is determined using the production function. The first order conditions of the profit maximization problem for the four inputs are as follows:

$$\frac{\partial p}{\partial x_1} = P(a_1 + 2a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4) - w_1$$
$$\frac{\partial p}{\partial x_2} = P(a_2 + 2a_{22}x_2 + a_{12}x_1 + a_{23}x_3 + a_{24}x_4) - w_2$$
$$\frac{\partial p}{\partial x_3} = P(a_{13} + 2a_{33}x_3 + a_{13}x_1 + a_{23}x_2 + a_{34}x_4) - w_3$$
$$\frac{\partial p}{\partial x_4} = P(a_4 + 2a_{44}x_4 + a_{14}x_1 + a_{24}x_2 + a_{34}x_3) - w_4$$

Given input and output prices, the system of four equations (one for each input first order condition) are solved simultaneously. Many software packages such as SHAZAM are proficient as solving such systems.

Prices are exogenous to the decision-maker under perfect competition. Since firms have no control over price, prices are randomly generated, and firms react to the given prices. Input and output prices were randomly generated, assuming a normal distribution, in SHAZAM. Once prices were generated, the system of first order conditions was solved and quantity values were obtained for each of the four inputs. After the input quantities were determined, the production function was used to calculate values for the output quantity.

This type of model specification is useful if one wishes to move out of perfect competition. For example, the profit maximization problem may be reformulated where price is endogenous to the producer. In this case, the same production function can be assumed and input prices may be randomly generated, and the firm can "choose" input quantities, which determines output, which then determines price (given a pre-determined demand curve).

Turning to the Hessian identities provided by Lau (1976), the Hessian of the production function can be used to determine the Hessian of the unrestricted profit function. To define the Hessian matrices, the variables x_1, x_2, x_3 and Y are considered variable commodities. These variables will collectively be referred to in net-put notation as y. For the purposes of the matrix manipulation, Y will represent the production function, R will represent the restricted profit function, U will represent the unrestricted profit function, p will represent commodity prices, x_4 will represent the fixed input, and q will represent the fixed input price. All prices are normalized on the variable input price of Y, or p.

Since we have assumed the coefficients for the production function (given above), we know the true Hessian matrix.

$$\begin{bmatrix} \frac{\partial^2 F}{\partial y^2} & \frac{\partial^2 F}{\partial y \partial x_4} \\ \frac{\partial^2 F}{\partial x_4 \partial y} & \frac{\partial^2 F}{\partial x_4{}^2} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix}$$

where x_4 is the fixed factor of production. Manipulating Lau's (1976) results indicates that the long-run Hessian typically generated by the unrestricted profit function may be equally obtained by inverting the Hessian for the production function.

$$\begin{bmatrix} \frac{\partial^2 U}{\partial p^2} & \frac{\partial^2 U}{\partial p \partial q} \\ \frac{\partial^2 U}{\partial q \partial p} & \frac{\partial^2 U}{\partial q^2} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 F}{\partial y^2} & \frac{\partial^2 F}{\partial y \partial k} \\ \frac{\partial^2 F}{\partial k \partial y} & \frac{\partial^2 F}{\partial k^2} \end{bmatrix}^{-1}$$

If the parameters of the estimation from the unrestricted profit function are denoted by $\$_{ii}$, then the following identity holds:

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{12} & b_{22} & b_{23} & b_{24} \\ b_{13} & b_{23} & b_{33} & b_{34} \\ b_{14} & b_{24} & b_{34} & b_{44} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix}^{-1}$$

In the short-run, some quantity information is often fixed. One can use short-run estimates obtained from the restricted profit function to recover long-run estimates. Let $(_{ii}$ represent the coefficients produced from the restricted profit function. Following Lau (1976), the conversion to the unrestricted profit function is:

let $\quad \Theta_1 = \left[\frac{\partial^2 R}{\partial x_4{}^2} \right]^{-1} = [g_{44}]$

let $\quad \Theta_2 = \left[\frac{\partial^2 R}{\partial x_4 \partial p} \right] = [g_{14} \quad g_{24} \quad g_{34}]$

let $\quad \Theta_3 = \left[\frac{\partial^2 R}{\partial p^2} \right] = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{12} & g_{22} & g_{23} \\ g_{13} & g_{23} & g_{33} \end{bmatrix}$

now,

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{12} & b_{22} & b_{23} & b_{24} \\ b_{13} & b_{23} & b_{33} & b_{34} \\ b_{14} & b_{24} & b_{34} & b_{44} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix}^{-1} = \begin{bmatrix} \left(\Theta_3 + \Theta_1' \cdot \Theta_2^{-1} \cdot \Theta_1\right) & \left(\Theta_2 \cdot \Theta_1\right)' \\ \left(\Theta_2 \cdot \Theta_1\right) & \Theta_1 \end{bmatrix}$$

Thus, there is a direct relation between the production function, the normalized unrestricted profit function, and the normalized restricted profit function.

To examine the effects of price variability, and measurement error on duality, several scenarios are considered. For each scenario, data are generated as indicated earlier, and two estimations are performed. Once the data are generated for a given level of price variability and measurement error (or noise), then the normalized restricted and unrestricted profit functions are estimated using the normalized quadratic functional form. The normalized quadratic form is chosen because it allows the Hessian values to be functions of parameter estimates only, and not depend on a particular data point. Lau (1976) indicated that the quadratic functional form was an logical choice when examining the results econometrically. Coefficient values from the unrestricted profit function can be directly compared to the inverse of the true parameters of the production function. Coefficients from the restricted profit function are manipulated as indicated above, and are then compared to the values from the unrestricted profit function. The values are converted to the long-run Hessian effects because economists are typically interested in examining the unrestricted effects.

The estimates are compared to the true values of the inverted production function Hessian indicate the ability of duality to recover the underlying “true” production technology. To quantify the comparison, a sum of squared error is calculated. The difference between the true value and the estimated values for each coefficient is squared and summed for the ten unique values of the Hessian matrix. For example if the long-run estimate from the restrited profit function was 1, the estimate from the unrestricted profit function was 1.5, and the true long-run value was 1.3, then for that coefficient, the squared error would be $(1-1.3)^2 + (1.5-1.3)^2 = 0.13$.

As previously indicated, the prices are generated with around a mean with a normal distribution. The coefficient of variation is used so that the percent variation can be held constant among prices. For example, all prices can be generated with a 30% coefficient of variation around the mean value. The same measure of variation is used to generate noise in the data. After the quantity variables are optimally determined, a percent variation is added to the mean values. Coefficient of variation for price variability was chosen at 10%, 30%, 50% because the values of 10% and 30% represented the low and high of actual production data over a 24 year time period from the Kansas Farm Management Data Bank.

Results

Empirically, Lau's (1976) Hessian identity results held (Table 1). It is important to note that the Hessian identities only perfectly hold when there is no noise (measurement error) in the data. In other words, any error introduced into the quantity or price variables considerably inhibits the performance of the dual relationship.

This dual relationship begins to deteriorate as noise is introduced into the system. Tables 2, 3, and 4 contain the results from the estimation of the systems with a 0.1%, 1%, and 5% coefficient of variation around the mean values of the quantity variables. For these results, the price variability is held constant at 30% so that the effect of measurement error may be identified. As can be seen by the examining the sum of squared errors from tables 2, 3, and 4, as the measurement error increases (or as more noise enters the system) in the quantity variables, the ability to recover the initial technology breaks down.

No only is measurement error important when examining the dual relationship, but also the relative price variability. For the technology to be recovered, there must be a large amount of relative price variability must occur. Results from the estimation of the systems with 10%, 30%

and 50% coefficient of variation around the mean values of the price variables are in tables 5, 2, and 6 respectively. The coefficient of variation around the quantity variables is held at 0.1% so that effect of price variability may be identified. As indicated by the sum of squared errors in these tables, and increase in relative price variability increases the performance of duality.

Table 8 summarizes the results of the preceding tables. Table 8 clearly indicated that a reduction in data measurement error and an increase in relative price variability increases the performance of the dual relationship. The magnitude of the sum of squared errors indicated that data noise plays a much larger role in duality than does price variability.

Conclusions and Implications

Duality has allowed researchers to recover production technology using several different approaches. Mathematically, several studies have proven the dual relationship between many relationships in economics. Economists use dual approaches to solve problems under the assumption that the same result may be achieved via another method.

Results of this study indicate that the dual relationship exhibited by the production function, the normalized restricted profit function, and the normalized unrestricted profit function are very sensitive to price variability and measurement error. In fact, a small amount of measurement error in the quantity variables causes the dual relationship to perform poorly. Perhaps researchers should approach dual problems with a bit more caution.

Table 1 – Comparison of Duality Results with a 0% Coefficient of Variation Around Quantities and a 30% Coefficient of Variation around Prices

	True Value	**Unrestricted Profit**	**Restricted Profit**	**Sum of Squared Error**
b_{11}	17.952	17.952	17.952	0
b_{22}	5.065	5.065	5.065	0
b_{33}	5.385	5.385	5.385	0
b_{44}	12.697	12.697	12.697	0
b_{12}	7.644	7.644	7.644	0
b_{13}	8.406	8.406	8.406	0
b_{14}	12.933	12.933	12.933	0
b_{23}	4.283	4.283	4.283	0
b_{24}	6.202	6.202	6.202	0
b_{34}	6.775	6.775	6.775	0
squared error				**0**

Table 2 – Comparison of Duality Results with a 0.1% Coefficient of Variation Around Quantities and a 30% Coefficient of Variation around Prices

	True Value	**Unrestricted Profit**	**Restricted Profit**	**Sum of Squared Error**
b_{11}	17.952	18.146	18.349	0.195
b_{22}	5.065	4.988	5.129	0.010
b_{33}	5.385	5.328	5.370	0.003
b_{44}	12.697	12.863	13.351	0.455
b_{12}	7.644	7.761	7.897	0.078
b_{13}	8.406	8.380	8.478	0.006
b_{14}	12.933	12.652	12.555	0.222
b_{23}	4.283	4.323	4.362	0.008
b_{24}	6.202	6.110	5.979	0.058
b_{34}	6.775	6.812	6.788	0.002
squared error				**1.037**

Table 3 – Comparison of Duality Results with a 1% Coefficient of Variation Around Quantities and a 30% Coefficient of Variation around Prices

	True Value	Unrestricted Profit	Restricted Profit	Sum of Squared Error
b_{11}	17.952	18.392	22.856	24.243
b_{22}	5.065	3.046	5.053	4.077
b_{33}	5.385	5.740	6.166	0.736
b_{44}	12.697	13.225	27.249	212.039
b_{12}	7.644	8.526	10.734	10.326
b_{13}	8.406	7.826	9.370	1.266
b_{14}	12.933	13.190	5.183	60.129
b_{23}	4.283	4.008	5.133	0.798
b_{24}	6.202	6.660	2.075	17.242
b_{34}	6.775	6.316	3.510	10.871
squared error				**341.726**

Table 4 – Comparison of Duality Results with a 5% Coefficient of Variation Around Quantities and a 30% Coefficient of Variation around Prices

	True Value	Unrestricted Profit	Restricted Profit	Sum of Squared Error
b_{11}	17.952	18.362	23.016	25.814
b_{22}	5.065	0.768	5.338	18.539
b_{33}	5.385	6.789	9.188	16.434
b_{44}	12.697	9.903	4.080	82.059
b_{12}	7.644	8.644	11.629	16.880
b_{13}	8.406	4.580	7.491	15.476
b_{14}	12.933	9.350	0.080	178.037
b_{23}	4.283	2.067	5.090	5.562
b_{24}	6.202	8.633	0.096	43.198
b_{34}	6.775	6.762	0.057	45.132
squared error				**447.131**

Table 5 – Comparison of Duality Results with a 0.1% Coefficient of Variation Around Quantities and a 10% Coefficient of Variation around Prices

	True Value	Unrestricted Profit	Restricted Profit	Sum of Squared Error
b_{11}	17.952	17.252	15.838	4.959
b_{22}	5.065	4.953	5.612	0.312
b_{33}	5.385	5.447	5.211	0.034
b_{44}	12.697	12.909	13.540	0.755
b_{12}	7.644	7.432	7.023	0.431
b_{13}	8.406	8.823	8.223	0.207
b_{14}	12.933	13.536	8.516	19.876
b_{23}	4.283	4.046	3.994	0.140
b_{24}	6.202	6.435	3.509	7.309
b_{34}	6.775	6.487	4.018	7.682
squared error				**41.706**

Table 6 – Comparison of Duality Results with a 0.1% Coefficient of Variation Around Quantities and a 50% Coefficient of Variation around Prices

	True Value	Unrestricted Profit	Restricted Profit	Sum of Squared Error
b_{11}	17.952	17.953	17.953	0.000
b_{22}	5.065	5.052	5.062	0.000
b_{33}	5.385	5.369	5.369	0.001
b_{44}	12.697	12.698	12.719	0.000
b_{12}	7.644	7.684	7.696	0.004
b_{13}	8.406	8.412	8.409	0.000
b_{14}	12.933	12.930	12.917	0.000
b_{23}	4.283	4.289	4.284	0.000
b_{24}	6.202	6.189	6.151	0.003
b_{34}	6.775	6.776	6.775	0.000
squared error				**0.009**

Table 7 – Comparison of Duality Results

Coefficient of Variation	Trial 1	Trial 2	Trial 3	Trial 4	Trial 5	Trial 6	Trial 7
Prices	30	30	30	30	10	30	50
Quantities	0	0.1	1	5	0.1	0.1	0.1
Sum of Squared Error	**0.00**	**1.04**	**341.72**	**447.13**	**41.71**	**1.04**	**0.00**

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