# Lifetime Leverage Choice for Proprietary Farmers in a Dynamic Stochastic Environment

## Robert A. Collins and Larry S. Karp

This article reviews various models that may be used to explain optimal lever-age choice for the proprietary farmer in a stochastic dynamic environment and develops a new model that highlights the risk of failure rather than the usual concept of risk as the variability of wealth. The model suggests that in addition to the usual factors, farm financial leverage is affected by age, wealth, and the opportunity cost of farming.

*Key words:* dynamic models, financial leverage, stochastic optimal control.

### Introduction

Farmers often make substantial changes in financial leverage over time. A frequently observed pattern is for a young farmer to start out heavily in debt, and pay down the debt over time, if the farm is successful. Young farmers who are unsuccessful seek alternative employment in the agribusiness sector, or join the rural-to-urban migration movement. Given constant parameters for risk aversion and the underlying probability distribution, systematic planned changes in leverage over time are not consistent with existing static models of farm leverage choice (such as Barry, Baker, and Sanint 1980, 1981; and Collins). In addition, many existing models of dynamic leverage choice either fail to explain this behavior, or are based on unattractive assumptions.

A popular explanation for these leverage changes is that young farmers are willing to take chances while older farmers are more conservative.[1] Even though changing risk aversion could explain changing lifetime leverage choices, there is little evidence that farmers become more risk averse with age.[2] An alternative explanation is that leverage choice is a dynamic process. This could mean that even if risk aversion or the parameters of the relevant density function did not change, a farmer could plan to change leverage over time.

This article introduces a stochastic optimal control model of farm leverage choice that models failure risk rather than the typical concept of variability of wealth. An analytic solution is presented that gives qualitative results on how optimal leverage changes over time and in response to changing wealth and the changing opportunity cost of being a farmer. The first section discusses previous dynamic stochastic models of proprietary leverage. The second section develops an alternative model of farm leverage choice that considers the farmer's risk of bankruptcy. The text discusses the assumptions and implications of the models, with most technical material left to appendices. The final section summarizes and states the primary testable implications of the model. The model suggests, in addition to the usual factors that affect the costs and benefits of leverage, that wealth, the opportunity cost of farming, and age also affect optimal leverage choice.

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## Review of Previous Models of Farm Leverage

Recent work by Featherstone, Preckel, and Baker uses discrete stochastic programming to evaluate leverage decisions. These models can evaluate various objectives including expected utility, allow detailed and realistic constraints, and can produce valuable managerial information. However, due to the curse of dimensionality, the model allows only a small number of time periods. The focus of the model is normative and shows only solutions for optimal initial debt rather than optimal behavior over time. Solutions for two time periods are shown in Featherstone, Baker, and Preckel, where optimal second-period debt is constant for low-risk aversion and decreasing for higher levels of risk aversion. The numerical methods used by Featherstone, Preckel, and Baker do not allow general conclusions about the qualitative nature of leverage decisions over time.

Lowenberg-DeBoer applies stochastic optimal control to the problem of farm leverage for the case of stochastic land values. He assumes that farm income is nonstochastic and that land prices follow a diffusion process. The farmer's objective is to maximize the expected present value of farm income and capital gains over a finite horizon. The resulting optimal time path for leverage decreases over time, but the optimal amount of leverage at any point in time increases with the drift parameter of the land price differential. Since the Lowenberg-DeBoer model assumes random land prices, constant farm income, and a risk-neutral decision maker, it emphasizes the farmer's role as a land speculator rather than as a business manager. It is interesting that the model implies that a risk-neutral person pays off debt over time; previous static models of farm leverage choice offer no explanation for this widely observed behavior.

Merton's model of lifetime portfolio choice also can be applied to the choice of farm debt. In that model, return on risky farm assets follows a diffusion process with drift parameter $a$ and variance $\sigma$. The return to assets could include both capital gains and farm income. The decision problem is to allocate wealth between risky farm assets with expected return $a$ and the riskless asset with known return $r$. By borrowing at rate $r$, total risky farm assets can exceed wealth.[3] The stochastic differential of wealth shows how changes in wealth ($W$) depend on the proportion of wealth invested in risky assets ($w$), consumption choices ($C$), and random elements:

$$dW = ([w(t)a + (1 - w(t)r)]W(t) - C(t))dt + w(t)\sigma Z(t)W(t)\sqrt{dt},$$

where $Z(t)$ is Brownian motion. The objective is to maximize the expected utility of discounted consumption,

$$\text{Max } E \int_0^T e^{-rt}U[C(t)] \, dt.$$

For the constant relative risk aversion utility function, $U(C) = C^\gamma/\gamma$, the solution to the maximization problem is

$$w^*(t) = \frac{a - r}{\sigma^2(1 - \gamma)}.$$

Several things are notable about this solution. First, it is not a function of either wealth or time. The optimal proportion of wealth invested in risky farm assets increases as the excess of the expected return over the borrowing rate increases, and it decreases with increases in risk and risk aversion, but the farmer would have no plans to change leverage over time, or as wealth changes. Second, this solution is the dynamic analog of some static models of farm leverage. Since equity is the equivalent of wealth, the proportion of wealth invested in risky assets can be regarded as the ratio of farm assets to farm equity. Therefore, the optimal ratio of equity ($E$) to assets ($A$) (or one minus the debt-asset ratio) is:

$$\left(\frac{E}{A}\right)^* = \frac{\sigma^2(1 - \gamma)}{a - r}.$$

This solution is analogous to Collins' equation (11) (p. 629), which shows an optimal equity-asset ratio of:

$$\left(\frac{E}{A}\right)^* = \frac{\rho\sigma^2}{R_a - k},$$

where $R_a$ is the expected return on risky assets, $k$ is the cost of borrowing, and $\rho$ is the coefficient of constant absolute risk aversion. The solutions are identical except that the coefficient of constant absolute risk aversion ($\rho$) from the negative exponential utility function replaces the coefficient of constant relative risk aversion ($1 - \gamma$) from the power utility function. This solution is also equivalent to equation (7) of Barry, Baker, and Sanint (1980, p. 153). Recognizing that their $\lambda$ is twice the coefficient of absolute risk aversion, and solving for the optimal ratio of equity to assets produces an identical result.

The above discussion suggests that these models neglect several important elements of the farm leverage decision. In addition, models based on a diffusion process implicitly assume that adjustments can be made at the same speed at which events occur. Given an equity level, a decrease in leverage requires that assets be sold and the proceeds used to retire debt. An increase in leverage requires an expansion in scale if equity is constant. Both of these adjustments require time and planning, and the farmer might simply be overtaken by events. Finally, none of these models explicitly consider the most severe potential consequence of leverage—bankruptcy.

## A Dynamic Leverage Model Based on a Jump Process

Dynamic leverage models based on a diffusion process assume that farm income or asset values vary continuously. While it is likely that farm asset values do vary continuously, and increases in leverage tend to amplify the effect of these variations on farm equity, it is less clear that these continuous variations are a primary concern to the farmer. An alternative objective is balancing the longer term returns from leverage with the hazard of a business failure caused by leverage. The model in this section shifts the focus from momentary variations in wealth to potential financial catastrophe. The farmer is risk neutral and wishes to maximize the expected present value of lifetime wealth. The choice variables are financial leverage and withdrawals of income for personal consumption over the lifetime of the farm. We assume that the only external financing available to the farmer is debt.

First, consider the relationship between the choice of leverage, $\delta$, and the rate of return on equity net of interest and taxes, $I(\delta)$.[4] As leverage increases from a low level, the rate of return on equity increases if the rate of return on assets exceeds the cost of borrowing, because more assets are working for each dollar of equity. This is the well-known leverage multiplier effect. Since increases in leverage also increase the probability of a disaster which causes increased risk of loss for the lender, the cost of borrowing also increases with leverage. Therefore, $I(\delta)$ increases at a decreasing rate, i.e., is concave in $\delta$. We also assume that the rate of return on equity is nonstochastic as long as the farm survives. The function $I(\;)$, however, could be regarded as the certainty equivalent of risky farm income. Appendix E addresses the problem when the return on equity is random, but the primary focus of the model is on the risk of failure rather than the variability of income.

The income produced by the farm depends on both the amount of equity and the rate of return on equity. Assuming constant returns to scale, farm income after interest and taxes at time $t$ is the rate of return on equity times the amount of equity, $I[\delta(t)]E(t)$. Defining $w(t)$ as the proportion of income withdrawn by the farmer for consumption, $1 - w(t)$ is the proportion of income that is retained. If the farm does not fail over the interval $dt$, the change in farm equity is $dE(t) = [1 - w(t)]I(\delta(t))]E(t)dt$. The rate of wealth accumulation depends on leverage and withdrawals for consumption.

The risk of financial catastrophe also depends on leverage choice. A farmer may face a variety of economic setbacks such as poor yields, low prices, floods, fires, or rapid
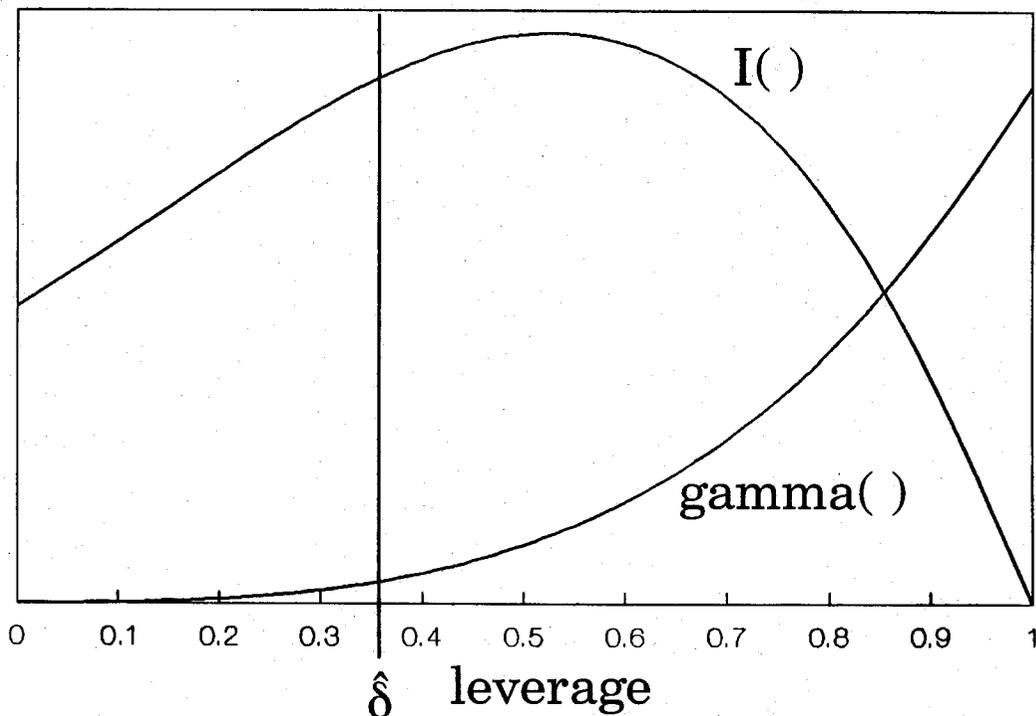
**Figure 1.** The rate of return on equity and the probability of failure as functions of leverage, and the maximum risk-adjusted rate of return on equity

declines in asset values. The ability of the farm to withstand negative income or other asset losses depends on leverage. If a farm loses 20% of its assets for any reason, a farm with a debt-asset ratio greater than 80% will be bankrupt. A farm with a lower leverage will survive. Therefore, given a distribution of different sizes of potential losses, the probability of a farm's failure is an increasing function of leverage. Figure 1 shows the probability of a disaster $\gamma(\ )$ and $I(\ )$. The leverage level that maximizes the difference between $I(\ )$ and $\gamma(\ )$ is denoted as $\hat{\delta}$. This is the leverage level at which the marginal effect of leverage on "return" equals the marginal effect of leverage on "risk," and the risk-adjusted rate of return on equity, $I(\ ) - \gamma(\ )$, is maximized.[5]

The stochastic differential for equity is the (deterministic) retention of earnings plus the stochastic term that indicates whether a disaster occurs:

$$(1) \qquad dE(t) = [1 - w(t)]I[\delta(t)]E(t)dt + Ed\pi,$$

where $Pr(d\pi = -1) = \gamma[\delta(t)]dt + o(dt)$, and $Pr(d\pi = 0) = 1 - \gamma[\delta(t)]dt + o(dt)$; $\gamma' > 0$, $\gamma'' > 0$; $o(dt)$ denotes terms of order $dt$. Hereafter, the variable $t$ is suppressed where the meaning is clear. The stochastic term is a jump process.[6] As long as $d\pi = 0$, equity continues to grow at the rate that earnings are retained. As leverage increases, however, the probability of a disaster increases. If a disaster occurs, $d\pi = -1$ and all equity disappears. This stochastic differential establishes the relationships between the farmer's choices and the equity of the farm. It serves as the constraint for the farmer's maximization problem.

The farmer's objective is maximization of the present value of expected wealth, the present value of the stream of withdrawals until retirement plus the present value of terminal equity resulting from the liquidation of the farm at retirement ($T$):

$$(2) \qquad J(E, t) = \max_{\substack{\delta \in [0, \ 1] \\ w \in [\underline{w}, \ \infty]}} \mathcal{E}_t\left[\int_t^T e^{-\rho s}wIEds + e^{-\rho T}E(T)\right],$$

where $\mathcal{E}_t$ is the expectations operator conditional on information available at time $t$ and $\rho$ is the riskless rate of time preference. The value function $J(\ )$ is discounted to time zero, the beginning of the planning period.

The farmer's choices are constrained. The debt-asset ratio must be between zero and one. Withdrawals have a lower bound of $\underline{w}$. A positive value of $\underline{w}$ means that the farmer cannot choose to retain all earnings, and absolute withdrawals must increase as equity increases. A negative value of $\underline{w}$ implies that the farmer has access to external equity to bring into the firm. In many cases it may be reasonable to assume that $\underline{w}$ is zero, i.e., the farmer can retain all earnings, but cannot obtain external equity financing. For the remainder of the text, we assume $w = 0$. The rate of withdrawals is unbounded above, which means the farmer can liquidate the farm at any time, and withdraw the equity, $E(t)$. This would entail an infinite withdrawal rate for an instant of time.

Maximization of (2) subject to (1) and the control constraints produces the optimal dynamic leverage and withdrawal strategy for the farm. Implicit in this statement of the problem is the assumption that if the farmer goes bankrupt, the value of the remaining time $(T - t)$ is zero. That is, the boundary condition is $J(0, t) = 0$. Under this assumption, the optimal plan solves the following dynamic programming equation (Mangel, pp. 50–51):

$$(3) \qquad -J_t = \max_{\delta, w} \ [e^{-\rho t} w I(\delta) E + J_E(1 - w) E I(\delta) - J\gamma(\delta)].$$

Subscripts indicate partial derivatives. The negative of the time derivative of the value function equals the maximized value of the sum of three terms. The first term is the present value of the flow of withdrawals; the second is the shadow value of equity, $J_E$, times the flow of retained earnings, conditional on not going bankrupt; the third is minus the value of the firm, $J(\ )$, times the probability of bankruptcy. The first two terms comprise the Hamiltonian of the deterministic version of this problem (zero bankruptcy risk).

The solution to this model is in appendix A. The primary implications of the solution are summarized in:

*Proposition 1*:

 (i)   If $I(\hat{\delta}) - \gamma(\hat{\delta}) = \rho$, it is optimal to set $\delta = \hat{\delta}$; withdrawal policy is irrelevant.
 (ii)  If $I(\hat{\delta}) - \gamma(\hat{\delta}) < \rho$, it is optimal to liquidate immediately.
 (iii) If $I(\hat{\delta}) - \gamma(\hat{\delta}) > \rho$, it is optimal to set $w = \underline{w}$. For $\underline{w} = 0$, it is optimal to maintain $\delta = \hat{\delta}$.

This proposition has a very intuitive interpretation. If the risk-adjusted rate of return on equity, $I(\hat{\delta}) - \gamma(\hat{\delta})$, equals the riskless discount rate, $\rho$, where the former is maximized, then the farmer is indifferent between liquidating and staying in business. If the maximized value of the risk-adjusted expected return is less than the discount rate, the farmer does better by immediate liquidation and investing the farm's equity at rate $\rho$. If the maximized risk-adjusted expected rate of return on equity is greater than the discount rate, the value of the farm as a going concern exceeds its liquidation value and, therefore, the farmer chooses to operate the farm. The farmer chooses the leverage that maximizes the risk-adjusted return on equity and holds it constant.

The ratio of the value of the firm as a going concern, $J(E, t)$, to its liquidation value, $E$, at time $t$ is denoted $f(t)$. We show in appendix A that if $\rho \leq I(\hat{\delta}) - \gamma(\hat{\delta})$, then $f(t) = \exp\{[\rho + \gamma(\hat{\delta}) - I(\hat{\delta})](t - T)\}$. The ratio of the value of the firm as a going concern to its liquidation value decreases over time at a constant rate equal to the difference between the risk-adjusted expected return and the riskless discount rate. Equivalently, $f(t)$ is the present value, discounted at $\rho$, of a dollar at time $t$ that is growing at rate $I(\hat{\delta})$ over the remaining horizon, times the probability that bankruptcy does not occur.

However, the boundary condition, $J(0, t) = 0$, ignores an important aspect of the farmer's choice problem: the opportunity cost of running the farm. The opportunity cost is a fixed

cost of being a farmer, but it does affect the optimal leverage choice, because the size of the capitalized alternative wage available off the farm affects the farmer's willingness to take a chance on bankruptcy. Thus, we now replace the assumption that conditional on bankruptcy, the value of the farmer's program (i.e., "wealth") is zero, with the assumption that the bankrupt farmer's wealth is the capitalized value of remaining wages available at the best off-farm opportunity, a measure of human capital. While financial capital is lost in a bankruptcy, human capital is retained. If the opportunity cost of farming[7] is $c per year, the present value of future wages at time $t$ for the remaining $T - t = \tau$ years is $c(1 - e^{-\rho\tau})/\rho$. Since the time path is being planned at time zero, this annuity of $c from $t$ to $T$ must be discounted back to time zero. Therefore, an alternative boundary condition is that the present value of the farmer's wealth at time zero if bankruptcy occurs at $t$ is:

$$J(0, t) = e^{-\rho t}(c/\rho)(1 - e^{-\rho\tau}).$$

The value of the farmer's remaining human capital from off-farm employment declines as time passes.

Two factors now influence the farmer's willingness to risk bankruptcy: the amount of equity that would be lost and the amount of capitalized wealth available from off-farm employment. The capitalized wealth available from off-farm employment declines over time, so if bankruptcy does occur, the farmer would prefer it to happen sooner rather than later. This causes risk to be allocated differently over time and, therefore, capital structure changes over time.

The opportunity cost of farming also affects the decision of whether to operate or liquidate the farm. When opportunity cost is ignored, the farmer will choose to operate the farm whenever the expected rate of return to equity ($I(\hat{\delta})$) exceeds the required rate of return for farming ($\rho + \gamma(\hat{\delta}) \equiv \alpha$). However, when opportunity cost is considered, the farmer may choose to liquidate the farm even if $I(\hat{\delta}) > \alpha$. For example, if the farmer's initial equity is very small, even the maximum possible rate of return on equity may not produce an earnings stream equivalent to the opportunity cost. Throughout the remainder of the article, we make two assumptions: (a) $I(\hat{\delta}) > \alpha$, and (b) when deciding whether to liquidate the farm at $t$, the farmer acts as if the option will not be available in the future. Assumption (a) simply means the problem is relevant, i.e., the risk-adjusted rate of return from farming exceeds the riskless rate. If that assumption did not hold, the farmer would want to liquidate for all levels of equity. Assumption (b) permits us to solve a simpler problem in which the value of the program at ($E, t$) is calculated using the optimal leverage, given that voluntary liquidation will never occur. This value is compared to the liquidating value at ($E, t$) to determine if liquidation is optimal.[8]

Figure 2 shows the following functions: The curve $\tilde{J}(E, t, c)$ gives the value of behaving optimally, conditional upon not liquidating. The linear function, $L(E, t, c)$ is the value of liquidating the farm at any time $t$ which is the capitalized value of remaining wages and current equity,

(4)
$$L(E, t, c) = e^{-\rho t}\left[\frac{c(1 - e^{-\rho\tau})}{\rho} + E\right].$$

As $E(0)$ approaches zero, $\tilde{J}(\ )$ also approaches zero while the liquidating value approaches the value of capitalized wages. Increases in $E(0)$ cause a linear increase in $L(\ )$, but nonlinear increases in $\tilde{J}(\ )$, if $I(\hat{\delta}) > \alpha$. The breakeven level of equity, $\bar{E}(t)$, equates the capitalized lifetime income stream from farming with the liquidating value of the farm. The breakeven level of equity is clearly a function of time since both functions shift over time.

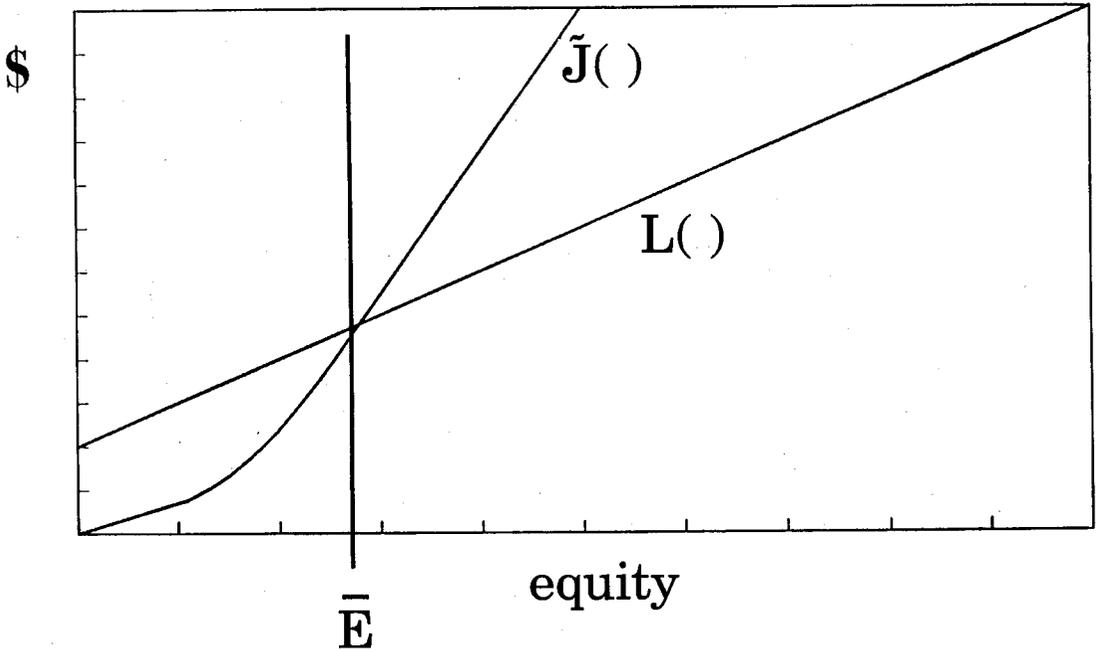Although it is not possible to obtain an exact solution of the control problem for $c >$

**Figure 2.** The value of operating the farm and the value of liquidating the farm as functions of equity, and the breakeven level of equity

0, an approximate solution is obtained by taking a Taylor expansion of the dynamic programming equation. Under the assumption that the value function and its derivatives are analytic in $c$ in the neighborhood of $c = 0$, this procedure gives decision rules which are approximately optimal for small values of $c$. The following discussion is based on analysis of these rules, and thus concerns the case where $c$ is small.

Designating $\tilde{E}(t)$ as the equity at $t$, given bankruptcy has not occurred, it is shown in appendix C that if $E(0) > \bar{E}(0)$, then $\tilde{E}(t) > \bar{E}(t)$ for all $t$. This means that if initial equity is large enough to make farming profitable compared to off-farm opportunities, it will remain so over time, if bankruptcy does not occur. Therefore, if the farmer chooses to farm initially, the option to liquidate prior to retirement has no value.

The next step is to determine an approximation, for small $c$, of the function $\tilde{J}(e, t, c)$ (the value of behaving optimally, conditional upon not liquidating). The solution provides the optimal debt-asset ratio, $\delta^*(E, t; c)$, and gives the minimum level of equity for farming to be profitable, $\bar{E}(t)$, when opportunity cost is positive.

Given the decision not to liquidate in the current instant and the result that liquidation will not occur in the future (so that it is optimal to set $w = 0$), the function $\tilde{J}$ satisfies the dynamic programming equation

(5) $\qquad -\tilde{J}_t = \max_{\delta}\{\gamma(\delta)[L(0, t, c) - \tilde{J}(E, t, c)] + J_E(E, t, c)I(\delta)E\}.$

Appendix B shows the derivation of the following first-order approximation and a summary of the results of a second-order approximation:

(6) $\qquad \tilde{J}(E, t, c) \approx e^{-\rho t}\left[f(t)E + \left(\frac{1 - e^{-\rho\tau}}{\rho} - \frac{1 - e^{-\alpha\tau}}{\alpha}\right)c\right]$

$\qquad\qquad\quad = e^{-\rho t}\left[f(t)E + c \int_t^T e^{-\rho(s-t)}(1 - e^{-\gamma(\delta)(s-t)})\, ds\right].$

This equation states that, to a first-order approximation, the increase in the value of equity due to a positive $c$ is the value of the discounted flow $c$ weighted by the probability, $1 - e^{-\gamma(\hat{\delta})(s-t)}$, that the farmer will go bankrupt and receive $c$.

The expression for $\bar{E}$, the critical level of equity, can be obtained by equating (6), the value of equity from farming, to the value of liquidating, $L(\ )$, given in (4).

$$(7) \qquad \bar{E}f(t) - \bar{E} = \frac{c}{\rho}(1 - e^{-\rho\tau}) - c \int_t^T e^{-\rho(s-t)}(1 - e^{-\gamma(\hat{\delta})(s-t)}) \, ds.$$

At the critical level of equity, the increase in the value of equity due to farming equals the difference between the certain wage (if liquidation occurs immediately) and the expected earnings from farming. Solving for $\bar{E}$ gives

$$(8) \qquad \bar{E}(t) = \frac{c}{\alpha}\left(\frac{1 - e^{-\alpha\tau}}{f(t) - 1}\right) = \frac{c}{\alpha}\left(\frac{e^{\alpha\tau} - 1}{e^{I(\hat{\delta})\tau} - e^{\alpha\tau}}\right).$$

Equation (8) provides a first-order approximation of the true $\bar{E}$; in order not to encumber the notation, $\bar{E}$ is used for both the true function and the approximation. Analysis of equation (8) shows how the breakeven level of equity changes over time:

$$(9) \qquad \frac{d\bar{E}}{d\tau} < 0,$$

$$(10) \qquad \lim_{\tau \to \infty} \bar{E}(\tau) = 0, \qquad \text{and}$$

$$(11) \qquad \lim_{\tau \to 0} \bar{E}(\tau) = \frac{c}{I(\hat{\delta}) - \rho - \gamma(\hat{\delta})}.$$

The minimum level of equity needed to persuade the proprietor to remain in farming increases as retirement approaches. For older farmers, the possibility of building up equity through farming is limited, but the risk of bankruptcy remains; they do better by liquidating the farm, investing in a riskless asset, and collecting the flow $c$. Rearranging (11) gives, at $\tau = 0$, $\bar{E}(I(\hat{\delta}) - \alpha) = c$; at retirement, a farmer with the critical level of equity receives a risk-adjusted flow from farming equal to the certain flow under the alternative employment.

For any value of equity, the return to liquidating at time zero is bounded above by $E + c/\rho$. However, $f$ becomes unbounded as $\tau$ approaches infinity, since $\alpha$ is positive; therefore, $\tilde{J}(\ )$ becomes unbounded for arbitrarily small positive initial levels of equity. That is, given a sufficiently long horizon, the value of remaining in farming exceeds $E + c/\rho$ for even very small values of equity. These results are summarized as:

> *Proposition 2:* Older farmers require higher levels of equity to induce them to remain in farming.

The next objective is to determine the effect of the opportunity cost of farming $(c)$ on the optimal leverage. When opportunity cost is not considered, optimal leverage $(\hat{\delta})$ simply maximizes the risk-adjusted rate of return to equity and is not affected by time or wealth. When opportunity cost is considered, optimal leverage is $\hat{\delta} + \Delta(E, c, t)$. The exact form of $\Delta$ is shown in appendix D. The increment to leverage $(\Delta)$ is a nonnegative function of age and wealth. Therefore, the consideration of opportunity cost creates a solution where more debt is optimal, but the size of the increment changes with age and wealth.

The partial effects of equity and time on optimal leverage are shown by

$$(12) \qquad \frac{\partial \Delta}{\partial E} = -\frac{\mu g c}{f E^2} < 0,$$

(13)
$$\frac{\partial \Delta}{\partial t} = \frac{\mu c e^{(I(\hat{\delta}) - \alpha)\tau}}{E f^2 \alpha} [\alpha - I(\hat{\delta})(1 - e^{\alpha\tau})].$$

The total effect of time when equity changes are considered is

(14)
$$\frac{d\Delta}{dt} = \frac{\partial \Delta}{\partial t} + \frac{\partial \Delta}{\partial E}\frac{dE}{dt} \leq \frac{\partial \Delta}{\partial t} - \frac{\mu g c}{f E^2} I(\hat{\delta}) E = \frac{\mu c e^{(I(\hat{\delta}) - \alpha)\tau}}{E f^2} < 0.$$

When opportunity cost is considered, the partial effect of an increase in equity makes farmers behave more cautiously despite risk neutrality. When equity is small, the farmer is nearly indifferent between going bankrupt and staying in business. This causes the farmer to take greater risks and choose higher leverage. When equity is large, a potential bankruptcy is much more costly and the return from alternative employment is small by comparison. These factors cause a wealthier farmer to behave more cautiously and choose lower leverage. This "risk-averse" behavior occurs even though the farmer is risk neutral.

The effect of age on leverage, holding equity constant, may be positive or negative. From (13), the leverage is decreasing over time as farmers near retirement ($\tau$ close to 0), but is increasing for young farmers ($\tau$ large). From the definition of $\Delta$ (appendix D), it can be seen that $\Delta = 0$ for $\tau = 0$ or $\tau = \infty$, and $\Delta > 0$ when $\tau$ is between zero and infinity. Therefore, given a constant level of equity, the model predicts relatively "risk-averse" behavior on the part of very old and very young farmers, despite the fact that all farmers are risk neutral.

If the horizon is sufficiently long, young farmers tend to be cautious because the value of farming is very large relative to the value of the nonfarm alternative [given assumption (a)]. As the horizon becomes shorter, the relative attractiveness of farming becomes less extreme, and leverage increases until it reaches a maximum. Beyond that point, (older) farmers tend to become less highly levered as they approach retirement; their opportunities outside farming decrease [$L(0, t, c)$ decreases], and they consequently decrease the risk of losing their wealth. In other words, the caution of young farmers is motivated by desire to remain in farming, whereas the caution of older farmers is motivated by the reluctance to depend on nonfarm income for retirement. The limiting leverage as $\tau$ approaches either zero or infinity is $\hat{\delta}$. Leverage is maximized where $\partial \Delta / \partial t = 0$ which, from (13), occurs at $\tau^* = -\ln[(I - \alpha)/I]/\alpha$. Suppose, for example, that $I(\hat{\delta}) = .077$ (continuously compounded) for an annual expected return on equity of 8%. If the required rate of return, $\alpha$, is 90% of $I$, then $t^* = 33$ years; that is, farmers within 33 years of retirement are expected to decrease their leverage given constant equity. If $\alpha$ is 80% of $I$, $t^* = 26$ years. Therefore, the theory suggests that the partial effect of age is consistent with a hump-shaped time profile of leverage for reasonable parameter values. An analysis of a cross-sectional data set should show a hump-shaped effect of age on leverage when equity is held constant.

The total effect of time on leverage considers both of the partial effects. A farmer expects equity to grow, conditional on not going bankrupt. As (14) shows, the effect of an increase in equity on leverage will always dominate the effect of time so that, for a particular farmer, leverage is expected to decrease over time. The inequality in (14) is due to the fact that at all points the leverage is not less than $\hat{\delta}$, so that the rate of increase in equity, given that bankruptcy does not occur, is not less than $I(\hat{\delta})$.

These conclusions are summarized as:

*Proposition 3:*
  (i)   Farmers with high equity choose lower debt-asset ratios than farmers with low equity.
  (ii)  Given the same level of equity, older farmers are less highly levered than middle-aged farmers; the latter may be more highly levered than very young farmers.
  (iii) An individual farmer tends to decrease leverage over time, conditional on not going bankrupt.

The use of "older," "middle-aged," and "very young" in Proposition 3 is imprecise, but the meaning is clear from the discussion of (13).

## Conclusions and Testable Implications

Static models of farm leverage choice which only consider how leverage amplifies ambient business risk, and dynamic models which ignore opportunity cost and bankruptcy risk neglect important factors in the leverage choice decision and fail to provide explanations for important observed phenomena. The model described above simultaneously provides an explanation for the apparent risk-seeking behavior some young farmers exhibit by carrying very heavy debt loads, and also the apparent risk-avoiding behavior that is frequently observed when middle-aged farmers choose to reduce their debt rather than expand the scale of the farm. However, these apparent risk-seeking and risk-avoiding behaviors arise from the model even though the farmer is risk neutral. Since a "risk response" arises from accounting for dynamics, opportunity cost, and stochastic failure when the utility function of wealth is linear, it suggests that there may be alternatives to the embattled expected utility hypothesis for explaining other risk-avoiding behaviors. The model also illustrates the potential importance that off-farm factors may have on optimal farm choices. While all introductory classes stress that an important cost of being a farmer is the opportunity cost, all too often this important factor is ignored in farm-level modeling.

The model has several assumptions and implications that could be subjected to econometric testing. The primary assumption is that farmers are not concerned about the effects of leverage on the variability of income, but are concerned about being devastated by a financial catastrophe. While this might be difficult to test conclusively, the secondary assumptions are more straightforward: Return on equity is a concave function of leverage but independent of scale, and the probability of failure is an increasing nonconcave (linear or convex) function of leverage.

The testable implications of the model are as follows:

(1) *Leverage depends on age.* The model predicts the partial effect of age to be hump-shaped. In a cross-section data set, if equity and opportunity cost are held constant, one would expect to find young farmers increasing leverage while older farmers are reducing leverage.

(2) *Leverage depends on wealth.* The partial effect of wealth is negative. If opportunity cost and age are held constant, one should find that high equity farmers choose lower debt-asset ratios than low equity farmers. This suggests, ceteris paribus, that the scale of the farm expands slower than equity.

(3) *Leverage depends on opportunity cost.* The partial effect of capitalized alternative earnings on leverage is positive. Clearly, opportunity cost depends on alternative earnings, time to retirement, and the discount rate.

(4) *Wealth affects the retirement decision.* The model suggests that richer farmers will retire *later*.

(5) *Surviving farms reduce debt over time.* The total effect of time, given other factors are allowed to adjust, is negative.

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## Notes

[1] This explanation was gained from conversations with agricultural bankers.
[2] Evidence on changes in risk aversion with age is not compelling. See Whittaker and Winter.
[3] Merton suggests this interpretation of the model in footnote 10.
[4] The income of the farm also depends on all the usual factors that affect the rate of return on assets. These other factors are assumed to be held constant. Therefore, $I(\ )$ represents the partial effect of leverage on the rate of return on farm equity.

[5] The expected return for $1 of equity for a period of $dt$ is $(1 + I)dt(1 - \gamma)dt = (1 + I - \gamma)dt - I\gamma dt^2$. Therefore, $I - \gamma$ may be regarded as a risk-adjusted rate of return.

[6] A standard method of analyzing this type of problem converts the stochastic control problem into a deterministic problem and then uses the Maximum Principle (Kamien and Schwartz). The nonstationarity due to the farmer's finite horizon makes that approach impractical here. The use of a jump process leads to a simple characterization of the solution.

[7] Other interpretations of this constant are possible. For example, it may reflect the nonpecuniary (dis)utility of living off the farm.

[8] Assumption (*b*) is innocuous in the following sense: If at an arbitrary time $t$ the farmer does not want to liquidate, then conditional upon not going bankrupt, there will be no desire to liquidate at any future time. This is discussed below and is proven for $t = 0$ in appendix C.

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## Appendix A: Solution to Control Model

Due to the assumptions on $I(\ )$ and $\gamma(\ )$ described in the text, the optimal value of $\delta$ lies in the interior of (0, 1). However, the maximand in (3) is linear in $w$, so the optimal value of this control may be on the boundary of the constraint. Liquidating the firm requires $w = \infty$; this is optimal if the value of the equity in the firm, $J(\ )$, is less than or equal to $e^{-\rho t}E$. Since liquidation is always an option, the minimum value of the firm is $e^{-\rho t}E$. If the value of equity is greater than the liquidation value, the manager retains the maximum amount of earnings by setting $w = \underline{w}$.

The first-order conditions for $\delta$ and $w$ are, respectively:

(A1)
$$[e^{-\rho t}w + J_E(1 - w)]EI' - J\gamma' = 0, \qquad \text{and}$$

(A2)
$$(e^{-\rho t} - J_E)EI \begin{cases} > 0 & \infty \\ = 0 \Rightarrow w = & ? \\ < 0 & \underline{w} \end{cases}.$$

The solution to the control problem requires finding the function $J(\ )$ such that when (A1) and (A2) are satisfied, both (3) and the boundary condition, $J(E, T) = e^{-\rho T}E$, are satisfied. It is straightforward to verify that the trial solution, $J(\ ) = f(t)e^{-\rho t}E$, with $f(T) = 1$, meets these conditions. The unknown function $f(t)$ gives the ratio of the value of the firm as a going concern to the liquidation value.

Substituting the trial solution into (3), (A1), and (A2) and simplifying gives

(A3)
$$\rho f - f = \max_{\delta, w} [(w + f - fw)I - f\gamma],$$

(A4)
$$I'(w + f - fw) = f\gamma',$$

(A5)
$$1 - f(t) \begin{cases} > 0 & \infty \\ = 0 \Rightarrow w = & ? \\ < 0 & \underline{w} \end{cases}.$$

(A dot over a variable represents total differentiation with respect to time.) Equation (A5) reproduces the intuition suggested above: If the value of the firm under optimal leverage is less than the liquidation value ($f(t) < 1$), then it is optimal to liquidate; if, under optimal leverage, the value is just equal to the liquidation value, withdrawal policies are irrelevant; and under the third case, the farmer wants to increase equity by as much as possible. Equation (A3) holds only for $f \geq 1$ since, if $f < 1$, the farmer liquidates and the problem ends.

## Proof of Proposition 1

From (A4) and (A5), the optimal values of $\delta$ and $w$ are functions of $f$, but do not depend explicitly on time. For $f \geq 1$, these functions can be substituted into (A3) to obtain the autonomous differential equation of the form $-\dot{f} = h(f)$; $dh/df$ is continuous for $f \geq 1$, where $dh(1)/df$ is defined as the limit of $dh/df$ as $f$ approaches 1, from above.

From the boundary condition on $J(\ )$, $f(T) = 1$. If the firm is still in business at $T$, (A4) implies $\delta[f(T)] = \hat{\delta}$. By the definition of $h(\ )$, $h(1) = I(\hat{\delta}) - \gamma(\hat{\delta}) - \rho$.

If $h(1) = 0$, then $f$ is stationary at $T$ and, therefore, $\dot{f}(t) \equiv 0$ and $f(t) \equiv 1$ for $t < T$. Part (i) of the proposition follows from (A4) and (A5).

If $h(1) < 0$ [$\rho > I(\hat{\delta}) - \gamma(\hat{\delta})$] and the firm had not liquidated by $T$, then $f(t) < 1$ for some $t$. By (A5), the firm must have liquidated by $T$. Suppose, contrary to part (ii) of the proposition, that $h(1) < 0$ and that it was not optimal to liquidate until time $t_1 > 0$. Then, over $(0, t_1)$ $f \geq 1$, where $h(\ )$ and $dh/df$ are continuous, and $f(t_1) \leq 1$. This implies $h(1) \geq 0$, a contradiction. It must be optimal to liquidate at time 0 if $h(1) < 0$. This is part (ii) of the proposition.

If $h(1) > 0$ [$\rho < I(\hat{\delta}) - \gamma(\hat{\delta})$], then $\dot{f}(T) < 0$. Therefore, $\dot{f}(t) < 0$ for finite $T - t$, so $f(t) > 1$ for $t < T$ (and finite $T - t$). From (A5), $w = \underline{w}$ is optimal. Rearrange (A4) to obtain:

$$(A6) \qquad \frac{\gamma'(\hat{\delta})}{I'(\hat{\delta})} = 1 + \underline{w}\left(\frac{1}{f} - 1\right).$$

For $\underline{w} = 0$, this gives part (iii) of the proposition.     Q.E.D.

## Appendix B: First- and Second-Order Approximations of the Dynamic Programming Equation

We assume that $\tilde{J}(\ )$ is analytic at $c = 0$, so that the following approximation is valid:

$$(6') \qquad \tilde{J}(E, t, c) = \tilde{J}(E, t, 0) + V(E, t)c + o(c)$$

$$\approx e^{-\rho t}\left\{f(t)E + (1 - e^{-\rho \tau})\frac{c}{\rho} + g(E, t)c\right\}.$$

The first equality gives the first-order approximation of $\tilde{J}(\ )$; the second equality uses the expression for $J(E, t)$ and decomposes the unknown function $V(\ )$ into $L(0, t, c) + g(\ )c$. Equation (6') [compare to equation (6) in text] states that the value of not liquidating is, to a first-order approximation, equal to the value of equity given that $c = 0$, plus the value of going bankrupt at $t$ given a positive level of $c$, plus the function $g(\ )c$. The last term must be negative since bankruptcy occurs with probability less than 1; that is, the function $g(\ )$ compensates for the fact that $L(0, t, c)$ overstates the value of the safety net.

Next, $\delta$ is replaced in equation (5) with the optimal function $\delta^*(E, t, c)$, and the maximization operator is removed. A Taylor expansion of the result at $c = 0$ is obtained, using equation (6') and the assumption $\underline{w} = 0$, which implies that $\delta^*(E, t, 0) = \hat{\delta}$, a constant. Using the envelope theorem then gives

$$(A7) \qquad \rho e^{-\rho t}\left[fE + (1 - e^{-\rho \tau})\frac{c}{\rho} + gc\right] - e^{-\rho t}[\dot{f}E - e^{-\rho \tau}c + \dot{g}_t c]$$

$$= -\gamma(\hat{\delta})e^{-\rho t}(fE + gc) + e^{-\rho t}(f + g_E c)EI(\hat{\delta}).$$

Equating coefficients in powers of $c$ implies

$$(A8) \qquad \dot{f} = [\alpha - I(\hat{\delta})]f; \qquad f(T) = 1,$$

and

$$(A9) \qquad 1 = g_t - \alpha g + EIg_E; \qquad g(T) = 0.$$

These equations use the definition $\alpha = \gamma(\hat{\delta}) + \rho$. Equation (A8) is a special case of (A3), using assumptions (a) and $\underline{w} = 0$. Equation (A9) has the solution

$$(A10) \qquad g = \frac{-(1 - e^{-\alpha \tau})}{\alpha},$$

which is independent of $E$. One can verify directly that (A10) satisfies (A9); the solution was obtained using the method of characteristics. Substituting (A10) into (6) gives the first-order approximation of $\tilde{J}(\ )$.

A higher order approximation was also considered. The second-order approximation of $\tilde{J}(\ )$, denoted $J^2(E, t, c)$, is

(A11)
$$J^2(\ ) = e^{-\rho t}\left\{f(t)E + \left[\frac{1 - e^{-\rho\tau}}{\rho} + g(t)\right]c + \frac{y(t)c^2}{2E}\right\},$$

where $y(t) < 0$ for $t < T$. Since $\tilde{J}(E, t, c)$ is bounded below by 0, $J^2$ underestimates $\tilde{J}$ for values of $E$ near 0. However, the farmer will not stay in business at low levels of equity. The first-order approximation of $\bar{E}$ underestimates the true value. A second-order approximation for optimal leverage was calculated. The expression is complicated, but a sufficient condition for $\partial^2\delta^*/\partial c^2 > 0$ is $\gamma'''(\hat{\delta}) - I'''(\hat{\delta}) \leq 0$; therefore, the first-order approximation of $\delta^*(\ )$ most likely understates optimal leverage.

## Appendix C: Proof that if $E(0) > \bar{E}(0)$, then $\tilde{E}(t) > \bar{E}(t)$

Since $\delta^* \geq \hat{\delta}$, $I > I(\hat{\delta})$ conditional on not going bankrupt, so

(A12)
$$\frac{\left[\dfrac{d\tilde{E}(t)}{dt}\right]}{\tilde{E}(t)} \geq I(\hat{\delta}).$$

For the remainder of the proof, the notation is simplified by writing $I(\hat{\delta})$ as $I$. Equation (8) implies

(A13)
$$\frac{\dfrac{d\bar{E}(t)}{dt}}{\bar{E}(t)} = \frac{Ie^{Ir} - \alpha e^{\alpha\tau}}{e^{Ir} - e^{\alpha\tau}} - \frac{\alpha e^{\alpha\tau}}{e^{\alpha\tau} - 1}.$$

The last two equations imply

(A14)
$$\frac{d(\ln \tilde{E}(t))}{dt} - \frac{d(\ln \bar{E}(t))}{dt} \geq \frac{Ie^{Ir} - Ie^{\alpha\tau} - (Ie^{Ir} - \alpha e^{\alpha\tau})}{e^{Ir} - e^{\alpha\tau}} + \frac{\alpha e^{\alpha\tau}}{e^{\alpha\tau} - 1}$$

$$= \frac{e^{\alpha\tau}}{(e^{Ir} - e^{\alpha\tau})(e^{\alpha\tau} - 1)}[(\alpha - I)(e^{\alpha\tau} - 1) + \alpha(e^{Ir} - e^{\alpha\tau})].$$

The term outside the square brackets is positive and the term inside the brackets simplifies to $I - \alpha + q(\tau)$, where $q(\tau) = \alpha e^{Ir} - Ie^{\alpha\tau}$, which is positive for $\tau > 0$.

Therefore, the rate of increase in $\tilde{E}$ is greater than that of $\bar{E}$, and the assertion is proven.     Q.E.D.

## Appendix D: Effect of Opportunity Cost on Optimal Leverage

Substituting the approximation of $\tilde{J}(\ )$, given by (6), into (5) implies that the approximation of $\delta^*(\ )$ must maximize:

(A15)
$$f(t)EI(\delta) - (f(t)E + g(t)c)\gamma(\delta).$$

The first-order condition to this problem is

(A16)
$$f(t)EI'(\delta) - (f(t)E + g(t)c)\gamma'(\delta) = 0.$$

Totally differentiating (A16) with respect to $c$ and evaluating the result at $c = 0$ gives

(A17)
$$\left.\frac{\partial\delta^*}{\partial c}\right|_{c=0} = \frac{\mu g(t)}{f(t)E},$$

with

(A18)
$$\mu \equiv \frac{\gamma'(\hat{\delta})}{I''(\hat{\delta}) - \gamma''(\hat{\delta})} < 0.$$

The optimal function $\delta^*$ can be approximated as

(A19)
$$\delta^*(E, t, c) = \hat{\delta} + \Delta(E, t, c) + o(c),$$

with

(A20)
$$\Delta \equiv \frac{\mu g(\tau)c}{f(\tau)E} \geq 0.$$

## Appendix E: Stochastic Income

The rate of return to equity could be modeled as stochastic, conditional on bankruptcy not occurring. This modification can be made by adding the product of Brownian motion and a function of $E$ and $\delta$ to equation (1). In the model where $c = 0$, the solution does not change. This can be seen from Ito's lemma and the fact that the value function remains linear in $E$. The risk-neutral farmer with $c = 0$ ignores a stochastic rate of return and concentrates on the risk of bankruptcy. Matters are more complicated where $c > 0$. The reason is that with a stochastic rate of return, assumption (b) is no longer innocuous; the relation described in appendix C cannot hold with probability 1. In addition, the value function is not linear in equity, so the instantaneous variance of the rate of return to equity would appear in the dynamic programming equation. The effect of including a stochastic return in the model with $c > 0$ remains an unresolved issue. The current model, which assumes a nonstochastic rate of return, is defended as a "certainty equivalent" approximation which is useful to highlight the farmer's response to the threat of catastrophic events.