"USE OF ASYMMETRIC-CYCLE AUTOREGRESSIVE MODELS TO IMPROVE FORECASTING OF AGRICULTURAL TIME SERIES VARIABLES"

by

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Threshold autoregressive (TAR) models were introduced to the econometrics literature by Tong (1983). In essence, in TAR models the autoregressive parameters are allowed to switch values over time as the residuals cross one or more thresholds. The most common TAR model exhibits two sets of parameters, which apply depending on the signs of the residuals. This implies that the characteristics, that is the height/dept and duration of the upward and downward cycles can be substantially different, i.e. that the cycles can be asymmetric. In the extreme, there could be upward but no downward cycling behavior or vice versa. In practice, there is no reason to expect that agricultural time series variables such as commodity cash or futures prices, crop acreage, etc., exhibit symmetric cycles as assumed in the standard autoregressive models. Therefore, TAR models could be very useful for analyzing the behavior of agricultural time-series variables.

Researchers have explored the use of TAR models in a variety of non-forecasting applications. Petruccelli and Woolford (1984) illustrate the estimation and use of a first order threshold autoregressive [TAR(1)] model. Tsay (1989) focuses on the testing for TAR processes, while Brockwell, Liu and Tweedie (1992) investigate the existence of stationary TAR moving average processes. Chang (1993) evaluates the consistency and limiting distribution of the least squares estimator of a TAR model. Balke and Fomby (1997) propose an approach for testing for cointegration in the presence of TAR rather than AR processes.

Recent applications of TAR models include Granger and Lee's (1989) investigation of production, sales and inventory relationships using multicointegration and non-symmetric error-correction models; Potter's (1995) analysis of the changes in real U.S. GNP; Bradley and Jansen's (1997) cross-country evaluation of business cycle dynamics; Obstfeld and Taylor's (1997) analysis purchasing power parity and the law of one price under imperfect arbitrage in the presence of transaction costs and uncertainty; and Goodwin and Piggott's (2001) evaluation of dairy price linkages among corn and soybean markets in North Carolina. Such research corroborates the potential importance of TAR models in the analysis of agricultural time series.

This study contributes to the understanding and application of TAR models by proposing and illustrating the use of a relatively straightforward Maximum Likelihood- (ML) based estimation procedure for cases when the conditional mean of the time series variable of interest is not zero but rather a function of one or more exogenous factors. A second contribution is to expand TAR models and estimation procedures to allow for the possibility of heteroskedasticity, i.e. for different levels of error term variation in upward versus downward cycles. A third major contribution of this study is to derive the formulas needed to obtain unbiased one-, two- and three-period-ahead predictions from TAR models. Substantial gains in forecasting precision are found when applying the proposed ML-based procedure to estimate TAR models of U.S. quarterly soybeans future prices and Brazilian coffee spot prices in comparison with AR models estimated using standard procedures. The estimated TAR models also provide useful insights on the markedly different dynamics of the upward versus the downward cycles exhibited by U.S. soybeans and Brazilian coffee prices.

Maximum Likelihood Estimation of TAR Models

The TAR model explored in this study is defined as follows:

(1) $y_t = \mathbf{x_t}\beta + e_t,$

$e_t = \phi_{1p}e_{t-1} + \phi_{2p}e_{t-2} + \ldots + \phi_{kp}e_{t-k} + v_t$ if $e_{t-1} \geq TR$, and

$e_t = \phi_{1n}e_{t-1} + \phi_{2n}e_{t-2} + \ldots + \phi_{kn}e_{t-k} + v_t$ if $e_{t-1} < TR$,

where y_t is the dependent variable of interest, $\mathbf{x_t}$ is a 1xm vector of exogenous variable values, β is an mx1 vector of intercept and slope parameters, v_t is a normally and independently distributed random variable with mean zero and variance σ^2, and TR is a threshold value to be estimated.

The TAR model above allows for two different autocorrelation regimes to apply depending on the value of the error term (e_t) during the previous time period. The occurrence of an error (e_{t-1}) greater than or equal to TR prompts the regime implied by the set of autocorrelation parameters $\boldsymbol{\phi_p} = [\phi_{1p}\ \phi_{2p} \ldots \phi_{kp}]$, while an error that is less than TR sets off the alternative regime implied by $\boldsymbol{\phi_n} = [\phi_{1n}\ \phi_{2n} \ldots \phi_{kn}]$. As previously discussed, in both theory and practice, this allows for an asymmetric cycling behavior of the error term.

More complicated TAR error-term structures where multiple thresholds are allowed have been explored in the econometrics literature (Balke and Fomby 1997; Enders and Granger 1998; Enders and Siklos 2001). Previous methodological research, however, has focused on the estimation and use of the pure time series form of equation (1), i.e. on the case where $\mathbf{x_t}\boldsymbol{\beta}=0$, which substantially facilitates estimation by permitting the use of Ordinary Least Squares (OLS) (Chan 1993). By extension, applications where $\mathbf{x_t}\boldsymbol{\beta}\neq 0$ have been simply estimated on the basis of the OLS residuals following a procedure that will be described later in this section.

The presentation of the proposed method begins with the concentrated log-likelihood function corresponding to the model in equation (1):

$$(2)\qquad -T\ln(\sigma)+\sum_{t=1}^{T} I_{tp}[-0.5T^{-1}\ln|\boldsymbol{\psi_p}|-0.5(e^*_{tp}/\sigma)^2]+\sum_{t=1}^{T} I_{tn}[-0.5T^{-1}\ln|\boldsymbol{\psi_n}| -0.5(e^*_{tn}/\sigma)^2]$$

where I_{tp} is an indicator variable taking a value of one if e_{t-1} greater than or equal to TR and a value of zero otherwise; I_{tn}=1-I_{tp}; $\boldsymbol{\psi_p}$ and $\boldsymbol{\psi_n}$ are the error term correlation matrices corresponding to the two possible autocorrelation regimes; and e^*_{tp} and e^*_{tn} are transformed residuals obtained as in the case of the standard autoregressive models (Judge et al. 1985, pp. 283-297):

$$(3)\qquad \mathbf{e^*_p} = \mathbf{P_p}(\mathbf{Y}-\mathbf{X}\boldsymbol{\beta})$$

$$(4)\qquad \mathbf{e^*_n} = \mathbf{P_n}(\mathbf{Y}-\mathbf{X}\boldsymbol{\beta})$$

where **Y** is the Tx1 vector of dependent variable observations; **X** is a Txm matrix of exogenous variable values; β is as previously defined; $\mathbf{P_p}$ and $\mathbf{P_n}$ are transformation matrices such that $(\mathbf{P'_p P_p})^{-1}=\psi_\mathbf{p}$ and $(\mathbf{P'_n P_n})^{-1}=\psi_\mathbf{n}$, respectively; and $\mathbf{e^*_p}$ and $\mathbf{e^*_n}$ are Tx1 vectors containing e^*_{tp} and e^*_{tn} (t=1,…T), respectively.

Note that the two components of the log-likelihood function {equation (2)} are the log-likelihoods corresponding to two standard autoregressive processes with error term correlation matrices $\psi_\mathbf{p}$ and $\psi_\mathbf{n}$. As in maximum likelihood estimation of standard autoregressive processes, $\psi_\mathbf{p}$, $\psi_\mathbf{n}$, $\mathbf{P_p}$, and $\mathbf{P_n}$, are functions of the autocorrelation parameters $\boldsymbol{\phi}_\mathbf{p} = [\phi_{1p}\ \phi_{2p} \ldots \phi_{kp}]$ and $\boldsymbol{\phi}_\mathbf{n} = [\phi_{1n}\ \phi_{2n} \ldots \phi_{kn}]$ (Judge et al. 1985, pp. 283-297). Also note that, if $\boldsymbol{\phi}_\mathbf{p} = \boldsymbol{\phi}_\mathbf{n}$, equation (2) becomes the log-likelihood function of a standard autoregressive process with error-term correlation matrix $\psi = \psi_\mathbf{p} = \psi_\mathbf{n}$. Thus, the null hypothesis of symmetric cycles (Ho: $\boldsymbol{\phi}_\mathbf{p} = \boldsymbol{\phi}_\mathbf{n}$) versus the alternative of asymmetric cycles (Ha: $\boldsymbol{\phi}_\mathbf{p} \neq \boldsymbol{\phi}_\mathbf{n}$) can be evaluated through a likelihood ratio test.

Also note that for the purposes of estimation the indicator variables I_{tp} and I_{tn} determine which of the two components of the log-likelihood function is switched on for the t^{th} observation, depending on the value of e_{t-1}. This switching process creates as discontinuity in the log-likelihood function with respect to the parameters in β. That is, in some regions of the β space a small change in one or more of the parameter values could cause at least one of the residuals ($e_{t-1} = y_{t-1}-\mathbf{x_{t-1}}\beta$) to transition from being below to being above TR, or vice versa, which would switch the values taken by the corresponding indicator variables I_{tp} and I_{tn}. Such a switch would apply a different autocorrelation regime to e_t causing a discrete shift in the log-likelihood function value. In addition, the log-likelihood function reaches a local maximum with respect to β for each set of values that can be taken by the pair of indicator variable vectors ($I_p = [I_{1p}, I_{2p},\ldots, I_{Tp}]$ and $I_n = [I_{1n}, I_{2n},\ldots, I_{Tn}] = 1-I_p = [1-I_{1p}, 1-I_{2p},\ldots, 1-I_{Tp}]$). Although this is not a violation of the regularity conditions for ML estimators (Judge et al. 1985, p. 178), after extensive testing it is concluded that in many cases this phenomenon makes it very difficult for gradient-based estimation algorithms such as Newton Raphson to smoothly converge into the function's global maximum.

To address the formerly described problem, this manuscript proposes the use of a grid search over the β and TR space combined with least-squares estimation of $\boldsymbol{\phi_p}$, $\boldsymbol{\phi_n}$ and σ, in order to maximize the log-likelihood function {equation (2)}. Without loss of generality, the steps for estimating a second-order TAR {TAR(2)} model with the proposed (TAR_P) procedure are:

a) Select the grid on the β and TR space over which the search is to be conducted.

b) For each set of β and TR values on the grid obtain the T*x*1 vector of residuals (r) and its corresponding set of T*x*1 indicator variable vectors (I_p and I_n; where, for t=2 to T I_{tp}=1 if $r_{t-1} \geq TR$ and I_{tp}=0 otherwise, I_{tn}= 1-I_{tp}, and I_{1p}= I_{1n}=0.5).

c) Also for each set of β and TR values on the grid obtain estimates of $\boldsymbol{\phi_p}$ and $\boldsymbol{\phi_n}$ by conducting OLS regressions of "indexed" residual vectors ($\mathbf{r^*_p}$ and $\mathbf{r^*_n}$) on their first and second lags. In the case of a TAR(2), the dependent variables in these regressions are the (T-2)*x*1 vectors $\mathbf{r^*_p}$ and $\mathbf{r^*_n}$, where $\mathbf{r^*_p}$= $\mathbf{I_p}$[3:T].*$\mathbf{r}$[3:T], $\mathbf{r^*_n}$= $\mathbf{I_n}$[3:T].*$\mathbf{r}$[3:T], .* is the element-by-element vector multiplication operator, and [3:T] indicates that only elements 3 to T of those vectors are included. The independent variable matrices are $\mathbf{rl^*_p}$=$\mathbf{I_p}$[3:T].*($\mathbf{r}$[2:T-1]~$\mathbf{r}$[1:T-2]) and $\mathbf{rl^*_n}$= $\mathbf{I_n}$[3:T].*($\mathbf{r}$[2:T-1]~$\mathbf{r}$[1:T-2]) where ~ is the horizontal vector concatenation operator.

d) Compute and add up the sum of the squared residuals (SSR) from the two OLS regressions in c) and obtain an estimate for σ^2 by dividing the SSR by T-2.

e) Use the so obtained estimates for β, TR, $\boldsymbol{\phi_p}$, $\boldsymbol{\phi_n}$, and σ^2 to compute the corresponding value of the log-likelihood function {equation (2)}. The ML-based set of estimates is of course the one that results in the highest log-likelihood value.

Note that this procedure only differs from a strict application of the maximum likelihood principle by the exclusion of the first two elements of the transformed error term vectors e^*_p and e^*_n. Therefore, it is asymptotically equivalent to maximum likelihood estimation, i.e. it will yield identical results given large sample sizes and fairly similar results in small sample applications.

Another contribution of this research is to consider the possibility that TAR error term variation might be different depending on whether the error is above or below TR, i.e. a heteroskedastic TAR specification. Under these circumstances the log-likelihood function includes two different variance parameters (σ_p and σ_n) as follows:

(5) $$\sum_{t=1}^{T} I_{tp}[-\ln(\sigma_p)-0.5T^{-1}\ln|\boldsymbol{\psi_p}|-0.5(e^*_{tp}/\sigma_p)^2]+\sum_{t=1}^{T} I_{tn}[-\ln(\sigma_n)-0.5T^{-1}\ln|\boldsymbol{\psi_n}| -0.5(e^*_{tn}/\sigma_n)^2]$$

where, for application of the proposed TAR_P method, σ^2_p and σ^2_n are estimated by:

(6) $s^2_p = SSR_p/\sum I_{tp}$, and

$s^2_n = SSR_n/\sum I_{tn}$,

where SSR_p and SSR_n are the sum of the squared residuals from the two OLS regressions in step c) and $\sum I_{tp}$ and $\sum I_{tn}$ represent the number on non-zero observations in those two regressions.

A simpler procedure to estimate TAR models (TAR_S) is also evaluated in this study. This method involves computing β and the corresponding residuals by OLS or by means of the more efficient AR model (note that the β estimates from these two procedures have the same expected value). The residuals (r_t) are then divided into two sets depending on whether $r_{t-1}\geq TR$ or $r_{t-1}<TR$ and the $\boldsymbol{\phi_p}$ and $\boldsymbol{\phi_n}$ parameter vectors are estimated by OLS regressions of r_t versus $r_{t-1}, r_{t-2},\ldots,r_{t-k}$ applied to each of these two sets of residuals. This process is repeated over a set of plausible TR values and the "optimal" TR and autoregressive parameter vector estimates are the ones corresponding to the lowest combined residual sum of squares from those two regressions.

Multi-Period TAR Forecasts

Unbiased multi-period forecasts from TAR models can be obtained by properly computing the expected values of the future errors conditional on the previous residuals. The forecasting formulas for a TAR(2) are derived next since the TAR(1) is a trivial case and the formulas corresponding to higher order processes are a logical extension of the TAR(2)'s. As in standard AR(2) models, the one-period-ahead forecast from a TAR(2) model is:

(7) $$y_{FT+1} = E[y_{T+1}|e_T,e_{T-1}] = E[\mathbf{x_{T+1}}\beta]+E[e_{T+1}|e_T,e_{T-1}] = \mathbf{x_{T+1}}E[\beta] + E[I_{Tp}\phi_{1p}e_T+I_{Tn}\phi_{1n}e_T] + E[I_{Tp}\phi_{2p}e_{T-1}+I_{Tn}\phi_{2n}e_{T-1}];$$

where the subscript F indicates forecast, T refers to the time period corresponding to the last available observation, and everything else is as defined before. Note that (7) is easily computed since e_T, I_{Tp} and I_{Tn} are known constants and $E[\beta]$, $E[\phi_{1p}]$, $E[\phi_{1n}]$, $E[\phi_{2p}]$, and $E[\phi_{2n}]$ can be replaced by ML estimates. The two-period-ahead forecast from a TAR(2) model is:

(8) $$y_{FT+2} = E[y_{T+2}|e_T,e_{T-1}] = E[\mathbf{x_{T+2}}\beta] + E[e_{T+2}] = \mathbf{x_{T+2}}E[\beta] + E[I_{T+1p}\phi_{1p}e_{T+1}+I_{T+1n}\phi_{1n}e_{T+1}] + E[I_{T+1p}\phi_{2p}e_T+I_{T+1n}\phi_{2n}e_T]$$

where (from here on) all expectations are conditional on the known e_T and e_{T-1} values and ϕ_{1p}, ϕ_{1n}, ϕ_{2p}, and ϕ_{2n} denote the ML estimates for these parameters, which are independent of all other random variables in the following equations. The last term of (8) is:

(9) $$E[I_{T+1p}\phi_{2p}e_T+I_{T+1n}\phi_{2n}e_T] = E[I_{T+1p}]\phi_{2p}e_T+E[I_{T+1n}]\phi_{2n}e_{T,}$$

where the expected values of the indicator variables at T+1 are computed as follows:

(10) $$E[I_{T+1p}] = Prob[e_{T+1}>0] = Prob[E[e_{T+1}]+v_{T+1}>0] = Prob[v_{T+1}>-E[e_{T+1}]] = Prob[v_{T+1}>-(I_{Tp}\phi_{1p}e_T+I_{Tn}\phi_{1n}e_T+I_{Tp}\phi_{2p}e_{T-1}+I_{Tn}\phi_{2n}e_{T-1})] = \int_C^{\infty} f_V\ ;$$

where $C =-(I_{Tp}\phi_{1p}e_T+I_{Tn}\phi_{1n}e_T+I_{Tp}\phi_{2p}e_{T-1}+I_{Tn}\phi_{2n}e_{T-1})$ is a known constant and fv is a normal density with mean zero and variance σ^2; and $E[I_{T+1n}]= Prob[e_{T+1}<0]= 1-Prob[e_{T+1}>0]= 1-E[I_{T+1p}]$.

The second term of (8) ($E[I_{T+1p}\phi_{1p}e_{T+1}+I_{T+1n}\phi_{1n}e_{T+1}]$) is more complicated to compute because it involves e_{T+1}. Specifically, substituting $E[e_{T+1}]+v_{T+1}$ for e_{T+1} yields:

(11) $$E[I_{T+1p}\phi_{1p}e_{T+1}+I_{T+1n}\phi_{1n}e_{T+1}] = E[I_{T+1p}\phi_{1p}(E[e_{T+1}]+v_{T+1}) + I_{T+1n}\phi_{1n}(E[e_{T+1}]+v_{T+1})] =$$

$$E[I_{T+1p}\phi_{1p}E[e_{T+1}] + I_{T+1n}\phi_{1n}E[e_{T+1}]] + E[I_{T+1p}\phi_{1p}v_{T+1} + I_{T+1n}\phi_{1n}v_{T+1}].$$

Since $E[e_{T+1}]$ is a known constant (defined as -C above), $E[I_{T+1p}\phi_{1p}E[e_{T+1}] + I_{T+1n}\phi_{1n}E[e_{T+1}]] = -C(\phi_{1p}E[I_{T+1p}] + \phi_{1n}E[I_{T+1n}])$, where $E[I_{T+1p}]$ and $E[I_{T+1n}]$ are computed as described in equation (10). Calculation of $E[I_{T+1p}\phi_{1p}v_{T+1} + I_{T+1n}\phi_{1n}v_{T+1}]$, on the other hand, requires knowledge of $E[I_{T+1p}v_{T+1}]$ and $E[I_{T+1n}v_{T+1}]$, which are obtained as follows:

(12) $$E[I_{T+1p}v_{T+1}] = \int_{C}^{\infty} v\, f_V ,$$

where C and fv are as defined above; and, since $E[I_{T+1n}v_{T+1}] + E[I_{T+1p}v_{T+1}] = E[v_{T+1}] = 0$, $E[I_{T+1n}v_{T+1}] = -E[I_{T+1p}v_{T+1}]$. Finally, the first term of (8) ($\mathbf{x_{T+2}}E[\beta]$) is obtained by replacing $E[\beta]$ with the MLE for β. The three-period-ahead forecast involves the following computations:

(13) $$y_{FT+3} = E[y_{T+3}|e_T,e_{T-1}] = E[x_{T+3}\beta] + E[e_{T+3}]$$

$$= x_{T+3}E[\beta] + E[I_{T+2p}\phi_{1p}e_{T+2}+I_{T+2n}\phi_{1n}e_{T+2}] + E[I_{T+2p}\phi_{2p}e_{T+1}+I_{T+2n}\phi_{2n}e_{T+1}]$$

The first term in (13) is obtained by replacing $E[\beta]$ with the MLE for β. Then, after substituting $E[e_{T+2}]+v_{T+2}$ and $E[e_{T+1}]+v_{T+1}$ for e_{T+2} and e_{T+1}, the second and third terms become:

(14) $$E[I_{T+2p}\phi_{1p}e_{T+2}+I_{T+2n}\phi_{1n}e_{T+2}] = E[I_{T+2p}\phi_{1p}E[e_{T+2}]] + E[I_{T+2n}\phi_{1n}E[e_{T+2}]] + E[I_{T+2p}\phi_{1p}v_{T+2}] + E[I_{T+2n}\phi_{1n}v_{T+2}],$$ and

(15) $$E[I_{T+2p}\phi_{2p}e_{T+1}+I_{T+2n}\phi_{2n}e_{T+1}] = E[I_{T+2p}\phi_{2p}E[e_{T+1}]] + E[I_{T+2n}\phi_{2n}E[e_{T+1}]] + E[I_{T+2p}\phi_{2p}v_{T+1}] + E[I_{T+2n}\phi_{2n}v_{T+1}].$$

Since $E[e_{T+1}]$ is a known constant (-C above), the terms in (15) are computed as follows:

(16) $$E[I_{T+2p}\phi_{2p}E[e_{T+1}]] = -C\phi_{2p}E[I_{T+2p}];$$

where $E[I_{T+2p}] = Prob[e_{T+2}>0] = Prob[E[e_{T+2}]+v_{T+2}>0] = Prob[v_{T+2}>-E[e_{T+2}]]$

$= Prob[v_{T+2}>-(I_{T+1p}\phi_{1p}e_{T+1}+I_{T+1n}\phi_{1n}e_{T+1}+I_{T+1p}\phi_{2p}e_T+I_{T+1n}\phi_{2n}e_T)]$

$= Prob[v_{T+2}>-(I_{T+1p}\phi_{1p}(E[e_{T+1}]+v_{T+1})+I_{T+1n}\phi_{1n}(E[e_{T+1}]+v_{T+1})+I_{T+1p}\phi_{2p}e_T+I_{T+1n}\phi_{2n}e_T)]$

$= \int_D^{\infty}\int_{-\infty}^{\infty} fv1v2$; where $D = -(I_{T+1p}\phi_{1p}(E[e_{T+1}]+v_{T+1})+I_{T+1n}\phi_{1n}(E[e_{T+1}]+v_{T+1})+I_{T+1p}\phi_{2p}e_T+ I_{T+1n}\phi_{2n}e_T)$,

$E[e_{T+1}] = -C$, $I_{T+1p} = 0.5(1+\{e_{T+1}/|e_{T+1}|\}) = 0.5(1+\{(E[e_{T+1}]+v_{T+1})/ |E[e_{T+1}]+v_{T+1}|\})$

$= 0.5(1+\{(-C+v_{T+1})/|-C+v_{T+1}|\})$, $I_{T+1n}=1-I_{T+1p}$, and fv_1v_2 is a bivariate normal density with means $[0,0]$ and variances $[\sigma^2,\sigma^2]$.

(17) $E[I_{T+2n}\phi_{2n}E[e_{T+1}]] = -C\phi_{2n}E[I_{T+2n}]$;

where $E[I_{T+2n}] = Prob[e_{T+2}<0] = 1-Prob[e_{T+2}>0] = 1-E[I_{T+2p}]$; and $E[I_{T+2p}]$ is computed as in (13).

(18) $E[I_{T+2p}\phi_{2p}v_{T+1}] = \phi_{2p}E[I_{T+2p}v_{T+1}] = \phi_{2p}\int_D^{\infty}\int_{-\infty}^{\infty} v1\, fv1v2$;

(19) $E[I_{T+2n}\phi_{2n}v_{T+1}] = \phi_{2n}E[I_{T+2n}v_{T+1}] = -\phi_{2n}E[I_{T+2p}v_{T+1}]$.

The last two terms in (14) ($E[I_{T+2p}\phi_{1p}v_{T+2}]$ and $E[I_{T+2n}\phi_{1n}v_{T+2}]$) are analogously computed:

(20) $E[I_{T+2p}\phi_{1p}v_{T+2}] = \phi_{1p}E[I_{T+2p}v_{T+2}]$; where $E[I_{T+2p}v_{T+2}] = \int_D^{\infty}\int_{-\infty}^{\infty} v2\, fv1v2$;

where $E[I_{T+2n}\phi_{1n}v_{T+2}] = \phi_{1n}E[I_{T+2n}v_{T+2}] = -\phi_{1n}E[I_{T+2p}v_{T+2}]$ and $E[I_{T+2p}v_{T+2}]$ is computed as above.

Lastly, computation of the first two terms in (13) is carried out as follows:

(21) $E[I_{T+2p}\phi_{1p}E[e_{T+2}]] = \phi_{1p}E[I_{T+2p}\{I_{T+1p}\phi_{1p}(E[e_{T+1}]+v_{T+1})+I_{T+1n}\phi_{1n}(E[e_{T+1}]+v_{T+1})+I_{T+1p}\phi_{2p}e_T$

$+ I_{T+1n}\phi_{2n}e_T\}]$, and

(22) $E[I_{T+2n}\phi_{1n}E[e_{T+2}]] = \phi_{1n}E[I_{T+2n}\{I_{T+1p}\phi_{1p}(E[e_{T+1}]+v_{T+1})+I_{T+1n}\phi_{1n}(E[e_{T+1}]+v_{T+1})+I_{T+1p}\phi_{2p}e_T$

$+ I_{T+1n}\phi_{2n}e_T\}]$.

Noting again that $E[e_{T+1}] = -C$, the known constant defined above, computation of (21) and (22) requires finding the expected value of products random variables such as:

(23) $E[I_{T+2p}I_{T+1p}] = \text{Prob}[e_{T+2}>0 \text{ and } e_{T+1}>0] = \text{Prob}[E[e_{T+2}]+v_{T+2}>0 \text{ and } E[e_{T+1}]+v_{T+1}>0]$

$$= \text{Prob}[v_{T+2}>-E[e_{T+2}] \text{ and } v_{T+1}>-E[e_{T+1}]] = \int_D^\infty \int_C^\infty f v1 v2\,;$$

(24) $E[I_{T+2p}I_{T+1p}v_{T+1}] = E[v_{T+1}|e_{T+2}>0, e_{T+1}>0] = E[v_{T+1}|E[e_{T+2}]+v_{T+2}>0, E[e_{T+1}]+v_{T+1}>0]$

$$= E[v_{T+1}|v_{T+2}>-E[e_{T+2}], v_{T+1}>-E[e_{T+1}]] = \int_D^\infty \int_C^\infty v1\, f v1 v2\,.$$

The expected values of the remaining products are computed analogously. The Gauss programs needed to compute these one-, two- and three-period ahead TAR model forecasts will be made available upon request.

Theoretical Performance

Monte Carlo experiments are conducted to evaluate the finite sample estimation and forecasting performance of the proposed ML-based estimation method (TAR_P) versus the simpler TAR_S, and the standard MLE for AR(1) and AR(2) models when the data-generating process is TAR. In these experiments $\mathbf{x_t}\beta=\beta_0+\beta_1 x_t=-1+1x_t$, x_t is a binomial random variable with P=0.5, and $\sigma=1$.

Table 1 shows the autocorrelation and partial autocorrelation functions (ACF/PACF) of the various TAR processes evaluated. Note that the general patterns of these ACF and PACF do not visually appear to be different from and could easily be confused with those of standard autoregressive processes. However, the specific arrays of error term autocorrelations implied by some TAR processes can not be closely approximated by standard autoregressive processes, which will later be shown to have substantial forecasting precision implications.

Figure 1 illustrates the dynamics of a typical TAR(1) error term with $\phi_{1p}=0.9$ and $\phi_{1n}=0$. Note that although the underlying random process (v_t) is white noise, nearly 80% of the simulated errors (e_t) are positive and the large sample average of all errors is approximately 1.35. The process can be described as fluctuating around zero (between –2 and 2) about 68% of the time and showing a marked upside cyclical behavior 32% of the time. A TAR(1) process with ϕ_{1p}=0.9 and ϕ_{1n}=-0.8 shows an average error term value of about 1.6, 85% positive errors, 63% within the –2 to 2 range, and 37% showing a marked upside cyclical behavior.

Interestingly, the transposing of a TAR(1) process parameters (i.e. to let $\phi_{1p}=\phi_{1n}$ and $\phi_{1n}=\phi_{1p}$) does not affect the ACF/PACF; it simple reverses the signs of the simulated error term patterns. In a TAR(1) with parameters ϕ_{1p}=–0.8 and ϕ_{1n}=0.9, for example, 85% of the errors are negative, rather than positive, the average of all errors is approximately -1.6, rather than 1.6, and 37% show a marked downside, rather than upside, cyclical behavior.

Figure 2 illustrates dynamics of a typical TAR(2) error term with parameters ϕ_{1p}=1.5, ϕ_{2p}=–0.8, ϕ_{1n}=–0.9 and ϕ_{2n}=0. Note that about 80% of the simulated errors are positive and the average of all errors is approximately 2.06. The behavior of this process can be described as randomly combining cycles of similar length that peak at different levels. In addition, although the error term drops below zero at the end of nearly every cycle, because ϕ_{1n}=–0.9, it rarely stays negative for more than one time period. Also, as in the case of a TAR(1), the transposing of the parameters of a TAR(2) process does not affect the ACF and PACF, but it does reverse the signs of the simulated error term patterns.

Table 2 shows select Monte Carlo simulation statistics about estimated TAR_P, TAR_S and standard AR models under different TAR(1) and TAR(2) processes and two sample sizes. The following conclusions are derived from those simulation statistics:

1) Although the TAR_P is a biased estimator for the intercept (β_0) and autocorrelation coefficients (ϕ_{1p}, ϕ_{2p}, ϕ_{1n} and ϕ_{2n}), it is a consistent estimator for all of those parameters. The

degree of bias decreases with sample size (T) and, in all cases evaluated, the percentage bias is negligible at T=500. The consistency of the proposed estimator is numerically verified by estimating the models with T=50000.

2) As OLS, the standard AR model estimation method is a biased and an inconsistent estimator for a TAR model's intercept. This is related to the previous discussion about the dynamics of TAR processes. Both OLS and the AR estimator are based on the assumption that the unconditional expected value of the error term is zero, and standard AR processes can only replicate symmetric error term cycles with expected values of zero. Therefore, when applied to the modeling of TAR processes with error terms exhibiting non-zero expected values, such as those depicted in figures 1 and 2, they do so through a biased/inconsistent estimation of the TAR processes' intercepts. In the case of the previously discussed TAR(1) with ϕ_{1p}=0.9 and ϕ_{1n}=-0.8, for example, OLS and a standard AR model would estimate the intercept with a bias and inconsistency of 1.6 (table 2), which is equal to the expected value of this particular TAR(1) error process. Biases in AR intercept estimation range from ±2.06 {TAR(2) with ϕ_{1p}=1.5, ϕ_{2p}=–0.8, ϕ_{1n}=–0.9 and ϕ_{2n}=0} to ±0.09 {TAR(2) with ϕ_{1p}=1.3, ϕ_{2p}=–0.6, ϕ_{1n}=0.5 and ϕ_{2n}=0.4}.

3) Both the TAR_P and the AR models are unbiased and consistent estimators for the slope parameter (β_1). However, as expected, the TAR_P is a more efficient estimator for this parameter. Estimation efficiency differences range from 7% to over 100%.

4) The TAR_P model forecasts obtained using the formulas derived in the previous section are unbiased both within and out of sample, although biased TAR_P model estimates for the intercept and autoregressive parameters have to be used to compute them. Interestingly, the predictions from the AR models, obtained using standard formulae, are unbiased as well.

5) Although the AR models can be used to approximate TAR processes, these approximations are generally far from perfect. The average R^2s of the AR models are 3% to 24% lower than those obtained when using the TAR_P models. Moderate to relatively high differences in forecasting precision, as measured by the root mean square of the within- and out-of-sample forecast errors, are also found between the estimated AR and TAR_P models when the underlying error term process is TAR. These differences range from about 3 to 65% and average approximately 20% for both the one- and the two-period-ahead out-of-sample forecasts, and are somewhat smaller in the case of the three-period-ahead predictions (table 2).

6) Since the simpler (TAR_S) method involves using the intercept estimate from the AR model which, as discussed in 2) above, is both biased and inconsistent, the TAR_S estimates for the autocorrelation parameters (ϕ_{1p}, ϕ_{2p}, ϕ_{1n}, and ϕ_{2n}) are also biased and inconsistent.

7) This biased and inconsistent estimation of the intercept and autocorrelation parameters by the TAR_S has substantial forecasting implications. The forecasting precision of the TAR_S is somewhere in between that of the standard AR and the proposed (TAR_P) estimation method. When the expected value of the underlying TAR error term process is substantially different from zero {such as in the TAR (2) with ϕ_{1p}=1.5, ϕ_{2p}=–0.8, ϕ_{1n}=–0.9 and ϕ_{2n}=0} the forecasting precision of the TAR_S is closer to that of the AR, while when the expected TAR error value is relatively close to zero {such as in the TAR (2) with ϕ_{1p}=1.3, ϕ_{2p}=–0.6, ϕ_{1n}=0.5 and ϕ_{2n}=0.4} the TAR_S forecasting precision is not substantially different from that of the TAR_P.

8) As expected, the differences in the root mean square of the one-period-ahead within sample forecasting errors corresponding to the AR, TAR_S and TAR_P models (WFE in table 2) are a good relative indicator of the differences in one-, two- and three-period-ahead out-of-sample forecasting precision (FE1, FE2 and FE3 in table 2) across these three models.

A final issue is whether TAR processes can be reasonable representations of the random components associated with some agricultural time series variables. To explore this issue, consider the most extreme of the previously discussed TAR processes where $y_t = \beta_0+\beta_1 x_t+e_t$; β_0=-1, β_1=1, and e_t follows a TAR with ϕ_{1p}=1.5, ϕ_{2p}=–0.8, ϕ_{1n}=–0.9 and ϕ_{2n}=0. If this model were estimated by OLS or by standard AR methods the intercept estimate would on average be 1.06 and the residuals ($y_t - \hat{B}_0 - \hat{B}_1 x_t$) would resemble those in figure 2, except that they will all be shifted by -2.06 and thus centered at an average of zero. Once centered at zero, the residuals in figure 2 do not appear very peculiar. In contrast, a properly estimated TAR model would on average have an intercept of -1 and residuals averaging 2.06 (as in figure 2). Yet, the unconditional expected value of the dependent variable predictions ($\hat{y}_t$) would be the same in both cases: $E[\hat{y}_t] = 1.06+x_t+E[e_t] = 1.06+x_t$ under OLS and $E[\hat{y}_t] = -1+x_t+E[e_t] = 1.06+x_t$ under the properly estimated TAR. Therefore, applied researchers should not be unsettled by the fact that TAR model residuals do not have an expected value of zero. TAR models simply provide for more complex, asymmetric cycling error term behaviors than standard AR models.

Applications

In this section, TAR_P, TAR_S and standard AR models of quarterly U.S. soybean future prices and of quarterly Brazilian coffee spot (New York) prices over the last three decades are estimated and compared. Both price series are stationary according to the augmented Dickey-Fuller unit root test (α<0.01). All initial models as specified with five autoregressive error term lags and a systematic component ($\mathbf{x_t \beta}$) consisting of an intercept and a simple linear time trend. The AR models are estimated using the standard Gauss 6.0 ARIMA procedure. The TAR_P and TAR_S are estimated using Gauss 6.0 code developed as part of this research. All Gauss programs used in these applications will be made available upon request.

In the case of coffee prices, the initial AR(5) and $TAR_P(5)$ models reach maximum log-likelihood function values of -411.34 and -377.02, respectively. Since the AR(5) is a restricted formulation of the $TAR_P(5)$, a likelihood ratio (LR) test for the statistical validity of those restrictions (Ho: $\boldsymbol{\phi_p}=\boldsymbol{\phi_n}$ and $\sigma^2_p=\sigma^2_n$ vs. Ha: $\boldsymbol{\phi_p}\neq\boldsymbol{\phi_n}$ and/or $\sigma^2_p\neq\sigma^2_n$) can be conducted. This test ($LR=2*(411.34-377.02)=68.64>\chi_{(6,\alpha=0.005)}=18.5$) strongly rejects those restrictions suggesting that the $TAR_P(5)$ is a statistically superior representation of the data-generating process.

The initial AR(5) model shows statistically insignificant fourth- and fifth-order autoregressive parameters ($\alpha=0.20$). A likelihood ratio test ($LR=2.57<\chi_{(2,\alpha=0.20)}=3.22$) does not reject the autoregressive parameter restrictions leading to the final AR(3) model presented in table 3. A Box-Pierce statistic of 19.68 does not reject the null hypothesis that the sample autocorrelation coefficients between the AR(3) model residuals and their first 20 lags are jointly equal to zero ($\alpha=0.25$), suggesting that these residuals are independently distributed.

The initial TAR_P model of Brazilian coffee prices shows several statistically insignificant autoregressive parameters ($\alpha=0.20$). A likelihood ratio test ($LR=3.30<\chi_{(5,\alpha=0.25)}=6.63$) does not reject the five autoregressive parameter restrictions leading to the final TAR_P model (table 3). A Box-Pierce statistic of 22.78 does not reject the null hypothesis that the sample autocorrelation coefficients between the final TAR_P model residuals and their first 20 lags are jointly equal to zero ($\alpha=0.25$), suggesting that these residuals are independently distributed as well.

With only three additional parameters, the final TAR_P model of Brazilian coffee prices exhibits a substantially higher maximum log-likelihood function value that the final AR model. The TAR_P model's R^2 of 0.894 is also noticeable higher than the AR's 0.838. The root mean square of the one-period-ahead within sample forecast error (RMSFE) is 19.60 cents/lb under the TAR_P versus 24.15 cents/lb, or 23.24% higher, under the AR (table 3). By any standards this would be considered a substantial difference in forecasting precision that justifies using the more sophisticated TAR_P modeling technique.

In addition, the TAR_P model provides useful insights into the dynamics of Brazilian coffee price cycles that are not attainable with the standard AR model. Specifically, 44% of the residuals are expected to be above the estimated threshold ($\hat{TR}$=-31.86), i.e. 44% of the price realizations are anticipated to be over PTR = $\mathbf{x_t}\hat{\beta}+\hat{TR}$ = 222.45-0.995t-31.86 = 190.59-0.995t. Since the expected value of the TAR error term in this case is $E[e_t]$=-31.58, the expected (unconditional) trend of Brazilian coffee prices is $E[y_t] = \mathbf{x_t}\hat{\beta}+E[e_t]$ = 222.45-0.995t-31.58 = 190.87-0.995t, i.e. coincidentally close to the previously discussed threshold equation (figure 3). This is markedly different from the 203.86-1.255t price trend implied by the final AR model.

In regard to the cycling behavior of Brazilian coffee prices, according to the final TAR_P model 44% of the price realizations will be above and 56% will be below PTR=190.59-0.995t. The dynamics of the upward cycles, however, are very different from those of the downward cycles (table 4). Only 7.58% of the prices crossing over PTR are expected not to be followed by at least one more price realization above that threshold equation, while the majority (21.84%) of the prices crossing under PTR will not be followed by additional price occurrences below the threshold. Interestingly, the AR implies that 19.81% of the prices crossing over or under this model's estimated trend equation will go back across the next quarter. On the other hand, the TAR_P model suggests that only 16% of the upward cycles will last between two and four quarters versus over 29% of the downward cycles; while 43% of the AR cycles are expected to be of this length. In contrast, nearly 45% of the upward cycles, but only 14% of the downward cycles (and 18% of the AR cycles), are predicted to last between five and seven quarters. And, while 27% of the downward cycles will tend to last over 10 quarters, less than 16% of the upward cycles (and about 10% of the AR cycles) are expected to be than long.

The σ_p and σ_n estimates {s_p and s_n in equation (6)} measure the root mean square of the one-period-ahead within sample forecast errors for the upward and downward cycles, respectively. Thus, another practical advantage of the TAR_P model is to be able to ascertain these differential levels of forecasting precision, that is, the typical prediction errors are estimated to be s_p=26.78 in the upward and s_n=14.35 cents/lb in the downward cycles. This indicates that the level of unpredictable variation in the upward price cycles is nearly twice as high as in the downward cycles; which is evident in the observed Brazilian coffee price data (figure 3).

In short, the TAR_P model suggests that the upward price cycles are expected to be moderately-lived but quickly reach fairly high levels. While about half of the downward cycles are expected to be short-lived (one to five quarters), the other half will tend to be moderately to longer- and could even be very long-lived and reach fairly low levels, but in a gradual manner.

According to the simulation results, the AR model yields biased intercept and unbiased but inefficient slope parameter estimates. In this case, the estimates for the both the intercept and the time trend parameter from the final AR(3) model (203.86 and -1.255) are markedly different from the TAR_P (222.45 and -0.995). This is not surprising since the TAR_P error term has a non-zero expected value ($E[e_t]$=-31.58). As a result, with an R^2 of 0.860 and WFE of 22.66 cents/lb (table 3), the TAR_S model's performance in this application is closer to the AR's (R^2=0.838 and WFE=24.15 cents/lb) than to the TAR_P's (R^2=0.894 and WFE=19.60 cents/lb).

In the case of U.S. soybean future prices, the initial AR(5) and TAR_P(5) models reach maximum log-likelihood function values of -516.87 and -499.30, respectively. A likelihood ratio test ($LR=2*(516.87-499.30)=35.14>\chi_{(6,\alpha=0.005)}=18.5$) strongly rejects the autoregressive parameter restrictions implied by the AR(5) model ($\boldsymbol{\phi_p}=\boldsymbol{\phi_n}$ and $\sigma^2_p=\sigma^2_n$) suggesting that the unrestricted TAR_P(5) model is statistically superior representation of the data-generating process.

The initial AR(5) model shows statistically insignificant third- fourth- and fifth-order autoregressive parameters (α=0.20). A likelihood ratio test ($LR=4.32<\chi_{(3,\alpha=0.20)}=4.64$) does not reject the restrictions leading to the final AR(2) model (table 3). A Box-Pierce statistic of 14.23 does not reject the null hypothesis independence in this model's residuals (α=0.25).

The initial TAR_P model of U.S. soybean prices only shows two statistically insignificant autoregressive parameters (α=0.20). A likelihood ratio test ($LR=2.24<\chi_{(2,\alpha=0.20)}=3.22$) does not reject the restrictions leading to the final TAR_P model (table 3). A Box-Pierce statistic of 19.76 does not reject the null hypothesis independence in this model's residuals either (α=0.25).

The final TAR_P model exhibits a markedly higher maximum log-likelihood function value than the final AR model (table 3). The TAR_P model's R^2 of 0.728 is substantially higher than the AR's 0.622 as well. The RMSFE is 50.86 cents/bushel under the TAR_P versus 59.93 cents/bushel, or 17.83% higher, under the AR. As in the case of the Brazilian coffee price models, such a difference in forecasting precision clearly justifies using the TAR_P technique.

Also as in the case of coffee, the TAR_P model provides valuable insights into the dynamics of soybean price cycles. Specifically, 41.6% of the price realizations are anticipated to be over $PTR= \mathbf{x_t}\hat{\beta}+\hat{T}R$ = 725.15-0.985t-9.18 = 715.97-0.985t. Since $E[e_t]$ = -23.27 in this case, the expected trend of U.S. soybean prices is $\mathbf{x_t}\hat{\beta}+E[e_t]$ = 725.15-0.985t-23.27 = 701.88-0.985t, i.e. somewhat lower than the threshold equation (figure 4). Note that, unlike in the case of coffee prices, the difference between the unconditional price expectations under the TAR_P and those implied by the final AR model (685.32-0.981t) is mainly due to the intercept estimate.

In regard to the cycling behavior of U.S. soybean prices, according to the final TAR_P model 41.6% of the price realizations will be above and 58.4% will be below PTR = 715.97-0.985t. As in the case of coffee prices, the dynamics of the upward cycles are very different from those of the downward cycles (table 4). The majority (61%) of the upward “cycles”, for example, are composed of only one or two price realizations above PTR, while just 32% of the downward “cycles” last two quarters or less. Interestingly, 42% of the upward and downward cycles implied by the estimated AR model are composed of one or two observations only. About 30% of the upward cycles and 38% of the downward cycles (and 41% of the AR cycles), are predicted to last between three and seven quarters. And, while 30% of the downward cycles would tend to last 8 quarters or more, less than 9% of the upward cycles (and about 17% of the AR cycles) are expected to be than long. Note that, as in the case of coffee prices, the cycling behavior implied by the TAR_P model appears to match the behavior of the observed soybean price data (figure 4).

The typical forecast errors are estimated to be s_p=48.94 in the upward and s_n=51.74 cents/lb in the downward cycles. That is, unlike in the coffee price model, unpredictable variation in the upward price cycles is nearly the same as in the downward cycles; which is evident in the observed soybean price data as well (figure 4).

In short, the estimated TAR_P model suggests that upward price cycles are substantially less likely than downward cycles lasting more than two quarters and, while most upward cycles exceeding two quarters are expected to be moderately-lived, the likelihood of both moderately and longer-lived downward cycles is substantial.

Finally, in this case, the estimates for the intercept and slope parameters from the final AR(2) model (685.32 and -0.981) are relatively much closer to the TAR_P’s (725.15 and -0.985) than in the Brazilian coffee price application. As a result, the TAR_S model’s R^2 (0.726) and RMSFE (51.22 cents/bushel) (table 3) are substantially closer to the TAR_P’s (0.728 and 50.86 cents/bushel) than to the AR’s (0.622 and 59.93 cents/bushel).

Concluding Remarks

The theoretical simulation and the empirical application results lead to three main conclusions: a) substantial gains in forecasting precision in relation to the standard AR models are expected by using the proposed (TAR_P) estimation method when the dependent variable is characterized by both a systematic and a random component and the random component follows a TAR rather than an AR process, b) the estimated TAR models also provide empirically valuable insights on the asymmetric dynamics of the upward and downward dependent variable cycles, c) since it is not possible to ascertain a priory whether the simpler TAR_S method will perform well in relation to the TAR_P in a particular application, the latter should be used in all cases. In short, researchers interested in thoroughly understanding the cycling behavior and obtaining more reliable forecasts for agricultural time series variables should consider using of the proposed procedure to ascertain if a TAR model is more suitable than a standard AR model in any particular application.

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Table 1. Correlation and Partial Autocorrelation Functions for TAR Processes Discussed and Evaluated in the Study

TAR Parameters		**TAR Parameters**		**TAR Parameters**		**TAR Parameters**		**TAR Parameters**		**TAR Parameters**		**TAR Parameters**	
ϕ_{1p}= 0.9 ϕ_{2p}= 0.0 ϕ_{1n}= 0.0 ϕ_{2n}= 0.0		ϕ_{1p}= 0.9 ϕ_{2p}= 0.0 ϕ_{1n}= -0.8 ϕ_{2n}= 0.0		ϕ_{1p}= 1.5 ϕ_{2p}= -0.8 ϕ_{1n}= 0.0 ϕ_{2n}= 0.0		ϕ_{1p}= 1.5 ϕ_{2p}= -0.8 ϕ_{1n}= -0.9 ϕ_{2n}= 0.0		ϕ_{1p}= 1.2 ϕ_{2p}= -0.8 ϕ_{1n}= 0.8 ϕ_{2n}= 0.0		ϕ_{1p}= 1.3 ϕ_{2p}= -0.6 ϕ_{1n}= 0.5 ϕ_{2n}= 0.4		ϕ_{1p}= 0.9 ϕ_{2p}= 0 ϕ_{1n}= -1.5 ϕ_{2n}= 0	
ACF	**PACF**	**ACF**	**PACF**	**ACF**	**PACF**	**ACF**	**ACF**	**ACF**	**PACF**	**ACF**	**PACF**	**ACF**	**PACF**
0.763	0.763	0.714	0.714	0.680	0.680	0.633	0.633	0.728	0.728	0.811	0.811	0.686	0.686
0.605	0.050	0.548	0.074	0.187	-0.513	0.068	-0.558	0.381	-0.321	0.626	-0.090	0.513	0.077
0.483	0.011	0.426	0.017	-0.170	-0.039	-0.285	0.002	0.148	0.036	0.470	-0.030	0.391	0.021
0.386	0.001	0.333	0.005	-0.273	0.033	-0.274	0.107	0.058	0.049	0.383	0.092	0.300	0.005
0.309	0.001	0.261	0.002	-0.174	0.037	-0.011	0.137	0.041	0.006	0.333	0.046	0.230	0.000
0.247	-0.004	0.206	-0.002	-0.013	0.011	0.247	0.077	0.040	-0.006	0.300	0.018	0.176	-0.004
0.197	-0.003	0.161	-0.003	0.089	-0.009	0.289	-0.031	0.032	0.000	0.267	0.006	0.134	-0.004
0.157	-0.004	0.124	-0.006	0.097	-0.007	0.136	0.007	0.020	-0.004	0.234	0.004	0.101	-0.005
0.126	-0.001	0.096	-0.001	0.047	0.005	-0.046	0.031	0.008	-0.003	0.204	0.006	0.075	-0.003
0.099	-0.006	0.073	-0.005	-0.009	-0.002	-0.126	-0.007	0.001	-0.004	0.177	-0.001	0.056	-0.004
0.078	0.000	0.055	-0.003	-0.037	-0.004	-0.081	-0.008	-0.002	0.000	0.155	0.003	0.041	-0.001
0.062	-0.003	0.040	-0.005	-0.034	-0.005	0.021	-0.001	-0.003	-0.004	0.136	-0.002	0.029	-0.006
0.048	-0.003	0.031	0.001	-0.015	-0.001	0.089	0.001	-0.004	-0.002	0.119	0.000	0.019	-0.004
0.037	-0.004	0.022	-0.006	0.002	-0.005	0.078	-0.009	-0.005	-0.005	0.104	-0.002	0.012	-0.004
0.027	-0.003	0.014	-0.004	0.009	-0.002	0.018	-0.001	-0.007	-0.002	0.090	-0.002	0.007	0.000
0.020	-0.002	0.007	-0.006	0.005	-0.006	-0.036	-0.004	-0.007	-0.005	0.078	-0.004	0.002	-0.006
0.014	-0.002	0.002	-0.002	-0.003	-0.003	-0.047	-0.003	-0.008	-0.003	0.066	-0.003	-0.002	-0.005
0.009	-0.006	-0.002	-0.005	-0.009	-0.003	-0.022	-0.005	-0.008	-0.004	0.057	-0.001	-0.004	-0.003

Note: The autocorrelation and partial autocorrelation functions (ACF and PACF, respectively) of 1000 simulated samples of size T=500 each are computed using the ACF and PACF commands in the ARIMA module of Gauss 6.0. Their averages over 1000 samples are reported in table 1.

Table 2. Select Monte Carlo Simulation Statistics About the Estimated TAR and AR Models Under Different TAR(1) and TAR(2) Processes and Sample Sizes

EM	T	AL	β_0	SE_1	ϕ_{1p}	ϕ_{2p}	ϕ_{1n}	ϕ_{2n}	R^2	WFE	FE1	FE2	FE3
TAR(1) Process with Parameters ϕ_{1p}= 0.9, ϕ_{2p}= 0.0, ϕ_{1n}= 0.0, ϕ_{2n}= 0.0													
AR	L	-0.53	0.35	0.076	0.76	--	0.76	--	0.62	1.03	1.04	1.28	1.42
TAR_P	L	-0.50	-0.98	0.071	0.89	--	-0.02	--	0.65	1.00	1.00	1.24	1.39
TAR_S	L	-0.53	0.35	0.076	0.83	--	0.68	--	0.63	1.03	1.03	1.27	1.41
AR	S	-0.52	0.35	0.174	0.73	--	0.73	--	0.59	1.02	1.06	1.31	1.45
TAR_P	S	-0.48	-0.89	0.166	0.86	--	-0.12	--	0.62	0.98	1.03	1.28	1.42
TAR_S	S	-0.51	0.35	0.174	0.79	--	0.65	--	0.59	1.01	1.05	1.30	1.44
TAR(1) Process with Parameters ϕ_{1p}= 0.9, ϕ_{2p}= 0.0, ϕ_{1n}= -0.8, ϕ_{2n}= 0.0													
EM	**T**	**AL**	**β_0**	**SE_1**	**ϕ_{1p}**	**ϕ_{2p}**	**ϕ_{1n}**	**ϕ_{2n}**	**R^2**	**WFE**	**FE1**	**FE2**	**FE3**
AR	L	-0.58	0.62	0.082	0.72	--	0.72	--	0.56	1.09	1.10	1.32	1.43
TAR_P	L	-0.50	-0.99	0.069	0.90	--	-0.80	--	0.64	1.00	1.00	1.26	1.39
TAR_S	L	-0.57	0.62	0.082	0.82	--	0.59	--	0.57	1.07	1.08	1.29	1.40
AR	S	-0.56	0.61	0.188	0.68	--	0.68	--	0.53	1.06	1.11	1.33	1.45
TAR_P	S	-0.49	-0.92	0.163	0.87	--	-0.69	--	0.62	0.98	1.03	1.28	1.42
TAR_S	S	-0.55	0.61	0.188	0.77	--	0.56	--	0.54	1.05	1.09	1.30	1.43
TAR(2) Process with Parameters ϕ_{1p}= 1.2, ϕ_{2p}= -0.8, ϕ_{1n}= 0.8, ϕ_{2n}= 0.0													
EM	**T**	**AL**	**β_0**	**SE_1**	**ϕ_{1p}**	**ϕ_{2p}**	**ϕ_{1n}**	**ϕ_{2n}**	**R^2**	**WFE**	**FE1**	**FE2**	**FE3**
AR	L	-0.59	-1.35	0.068	0.97	-0.33	0.97	-0.33	0.62	1.10	1.13	1.58	1.72
TAR_P	L	-0.49	-1.00	0.057	1.20	-0.80	0.79	0.00	0.69	0.99	1.03	1.47	1.64
TAR_S	L	-0.52	-1.35	0.068	1.19	-0.67	0.76	-0.01	0.67	1.02	1.05	1.49	1.65
AR	S	-0.57	-1.35	0.158	0.96	-0.34	0.96	-0.34	0.61	1.07	1.12	1.60	1.76
TAR_P	S	-0.47	-1.01	0.142	1.17	-0.80	0.77	0.00	0.69	0.96	1.04	1.50	1.67
TAR_S	S	-0.50	-1.35	0.158	1.17	-0.66	0.76	-0.04	0.66	1.00	1.06	1.52	1.70
TAR(2) Process with Parameters ϕ_{1p}= 1.5, ϕ_{2p}= -0.8, ϕ_{1n}= 0.0, ϕ_{2n}= 0.0													
EM	**T**	**AL**	**β_0**	**SE_1**	**ϕ_{1p}**	**ϕ_{2p}**	**ϕ_{1n}**	**ϕ_{2n}**	**R^2**	**WFE**	**FE1**	**FE2**	**FE3**
AR	L	-0.71	0.28	0.075	1.03	-0.52	1.03	-0.52	0.63	1.24	1.25	1.80	1.96
TAR_P	L	-0.50	-1.00	0.049	1.50	-0.80	0.00	0.00	0.76	0.99	1.01	1.60	1.88
TAR_S	L	-0.65	0.28	0.075	1.35	-0.78	0.76	-0.33	0.67	1.16	1.16	1.71	1.94
AR	S	-0.69	0.28	0.173	1.02	-0.52	1.02	-0.52	0.63	1.21	1.29	1.84	1.98
TAR_P	S	-0.48	-0.98	0.121	1.48	-0.79	-0.02	0.01	0.76	0.96	1.07	1.64	1.91
TAR_S	S	-0.63	0.28	0.173	1.32	-0.77	0.77	-0.34	0.67	1.13	1.19	1.76	1.98
TAR(2) Process with Parameters ϕ1p= 1.3, ϕ2p= -0.6, ϕ1n= 0.5, ϕ2n= 0.4													
EM	**T**	**AL**	**β_0**	**SE_1**	**ϕ_{1p}**	**ϕ_{2p}**	**ϕ_{1n}**	**ϕ_{2n}**	**R^2**	**WFE**	**FE1**	**FE2**	**FE3**
AR	L	-0.64	-1.09	0.077	0.89	-0.09	0.89	-0.09	0.69	1.15	1.17	1.57	1.78
TAR_P	L	-0.49	-1.00	0.056	1.30	-0.60	0.49	0.40	0.77	0.99	1.02	1.42	1.64
TAR_S	L	-0.52	-1.09	0.077	1.25	-0.54	0.52	0.36	0.75	1.02	1.04	1.45	1.67
AR	S	-0.61	-1.07	0.176	0.87	-0.11	0.87	-0.11	0.66	1.11	1.18	1.59	1.82
TAR_P	S	-0.47	-0.99	0.144	1.29	-0.62	0.46	0.38	0.76	0.96	1.05	1.47	1.70
TAR_S	S	-0.52	-1.07	0.176	1.18	-0.49	0.56	0.25	0.72	1.02	1.09	1.50	1.73

Table 2 (continued). Select Monte Carlo Simulation Statistics About the Estimated TAR and AR Models Under Different TAR(1) and TAR(2) Processes and Sample Sizes

TAR(2) Process with Parameters ϕ_{1p}= 1.5, ϕ_{2p}= -0.8, ϕ_{1n}= -0.9, ϕ_{2n}= 0													
EM	**T**	**AL**	β_0	**SE$_1$**	ϕ_{1p}	ϕ_{2p}	ϕ_{1n}	ϕ_{2n}	**R^2**	**WFE**	**FE1**	**FE2**	**FE3**
AR	L	-0.98	1.06	0.099	0.99	-0.56	0.99	-0.56	0.61	1.62	1.65	2.33	2.45
TAR$_P$	L	-0.50	-1.00	0.048	1.50	-0.80	-0.89	0.01	0.85	0.99	1.01	1.66	1.99
TAR$_S$	L	-0.89	1.06	0.099	1.32	-0.77	0.64	-0.38	0.68	1.47	1.43	2.06	2.34
AR	S	-0.94	1.06	0.229	0.98	-0.57	0.98	-0.57	0.61	1.57	1.69	2.41	2.50
TAR$_P$	S	-0.48	-0.98	0.115	1.49	-0.79	-0.85	-0.01	0.84	0.97	1.06	1.69	2.04
TAR$_S$	S	-0.86	1.06	0.229	1.29	-0.76	0.66	-0.41	0.67	1.43	1.49	2.16	2.43

Notes: The statistics are over 10000 models estimated on the basis of a similar number of simulated samples. EM refers to the type of model being estimated: AR is the standard autoregressive model; TAR$_P$ is a TAR model estimated using the proposed ML-based method; and TAR$_S$ is a TAR model estimated on the basis of the AR residuals. T is the sample size {L=large (T=500), and S=small (T=100)}. AL indicates the average maximum value reached by the corresponding mean log-likelihood function. β_0, ϕ_{1p}, ϕ_{2p}, ϕ_{1n}, and ϕ_{2n} refer to the averages of the estimates for the intercept and the four autocorrelation process parameters, respectively. SE$_1$ stands for the standard deviation of the 10000 slope parameter estimates. The R^2 is computed as the square of the correlation coefficient between the within sample one-period-ahead autoregressive predictions and the actual dependent variable values. WFE stands for the averages the within sample root mean square errors of the one-period ahead forecasts; and FE1, FE2, and FE3 refer to the root mean square errors of the one-, two- and three-period-ahead out of sample forecasts.

Table 3. Key Statistics of Models for Quarterly Brazilian Coffee Spot and U.S. Soybean Future Prices

Final AR Model of Coffee Prices												
PR	β_0	β_1	ϕ_{1p}	ϕ_{2p}	ϕ_{3p}	ϕ_{4p}	ϕ_{5p}	ϕ_{1n}	ϕ_{2n}	ϕ_{3n}	ϕ_{4n}	ϕ_{5n}
PE	203.86	-1.255	1.136	-0.547	0.183	0.000	0.000	1.136	-0.547	0.183	0.000	0.000
SE	20.58	0.293	0.095	0.135	0.095	--	--	0.095	0.135	0.095	--	--
PV	0.000	0.000	0.000	0.000	0.056	--	--	0.000	0.000	0.056	--	--
	MLFV=-412.63					WFE=24.15				R^2=0.838		
Final TAR_P Model of Coffee Prices												
PR	β_0	β_1	ϕ_{1p}	ϕ_{2p}	ϕ_{3p}	ϕ_{4p}	ϕ_{5p}	ϕ_{1n}	ϕ_{2n}	ϕ_{3n}	ϕ_{4n}	ϕ_{5n}
PE	222.45	-0.995	0.776	-0.743	0.428	-0.488	0.000	0.953	0.000	0.000	0.000	0.000
SE	--	--	0.154	0.190	0.195	0.134	--	0.027	--	--	--	--
PV	--	--	0.000	0.000	0.031	0.000	--	0.000	--	--	--	--
	MLFV=-378.67					WFE=19.60				R^2=0.894		
Final TAR_S Model of Coffee Prices												
PR	β_0	β_1	ϕ_{1p}	ϕ_{2p}	ϕ_{3p}	ϕ_{4p}	ϕ_{5p}	ϕ_{1n}	ϕ_{2n}	ϕ_{3n}	ϕ_{4n}	ϕ_{5n}
PE	203.86	-1.255	1.222	-0.833	0.569	-0.357	0.000	0.872	0.000	0.000	0.000	0.000
SE	--	--	0.132	0.200	0.203	0.138	--	0.050	--	--	--	--
PV	--	--	0.000	0.000	0.006	0.011	--	0.000	--	--	--	--
	MLFV=-391.06					WFE=22.66				R^2=0.860		
Final AR Model of Soybean Prices												
PR	β_0	β_1	ϕ_{1p}	ϕ_{2p}	ϕ_{3p}	ϕ_{4p}	ϕ_{5p}	ϕ_{1n}	ϕ_{2n}	ϕ_{3n}	ϕ_{4n}	ϕ_{5n}
PE	685.32	-0.981	0.899	-0.194	0.000	0.000	0.000	0.899	-0.194	0.000	0.000	0.000
SE	40.15	0.570	0.094	0.095	--	--	--	0.094	0.095	--	--	--
PV	0.000	0.088	0.000	0.043	--	--	--	0.000	0.043	--	--	--
	MLFV=-519.03					WFE=59.93				R^2=0.622		
Final TAR_P Model of Soybean Prices												
PR	β_0	β_1	ϕ_{1p}	ϕ_{2p}	ϕ_{3p}	ϕ_{4p}	ϕ_{5p}	ϕ_{1n}	ϕ_{2n}	ϕ_{3n}	ϕ_{4n}	ϕ_{5n}
PE	725.15	-0.985	0.749	0.000	0.461	-1.036	0.816	0.985	0.000	-0.280	0.333	-0.299
SE	--	--	0.109	--	0.145	0.176	0.139	0.080	--	0.105	0.125	0.089
PV	--	--	0.000	--	0.002	0.000	0.000	0.000	--	0.009	0.009	0.001
	MLFV=-500.42					WFE=50.86				R^2=0.728		
Final TAR_S Model of Soybean Prices												
PR	β_0	β_1	ϕ_{1p}	ϕ_{2p}	ϕ_{3p}	ϕ_{4p}	ϕ_{5p}	ϕ_{1n}	ϕ_{2n}	ϕ_{3n}	ϕ_{4n}	ϕ_{5n}
PE	685.32	-0.981	0.747	0.000	0.462	-1.035	0.808	0.892	0.000	-0.290	0.325	-0.277
SE	--	--	0.083	--	0.145	0.177	0.144	0.092	--	0.107	0.126	0.089
PV	--	--	0.000	--	0.002	0.000	0.000	0.000	--	0.008	0.011	0.003
	MLFV=-501.18					WFE=51.22				R^2=0.726		

Notes: PR, PE, SE and PV stand for parameter, parameter estimate, standard error estimate and p-value; MVLF is the maximum value reached by the log-likelihood function; WFE is the within sample root mean square error of the one-period ahead forecasts; and the R^2 is computed as described under table 2. Autoregressive parameters that are statistically insignificant at the 20% level have been set equal to zero.

Table 4. Relative Frequencies of the Cycle Durations Implied by the Estimated TAR_P and AR Models for Brazilian Coffee and U.S. Soybean Prices

Cycle Length (quarters)	TAR_P – Coffee Prices Upward	TAR_P – Coffee Prices Downward	AR–Coffee Prices	TAR_P – Soybean Prices Upward	TAR_P – Soybean Prices Downward	AR–Soybean Prices
1	7.58%	21.84%	19.81%	37.63%	21.20%	25.47%
2	3.51%	12.77%	18.39%	23.32%	10.75%	16.68%
3	4.61%	9.20%	14.58%	6.51%	12.59%	12.80%
4	7.92%	6.83%	10.34%	11.17%	5.00%	9.69%
5	13.57%	5.47%	7.46%	8.14%	6.63%	7.66%
6	18.73%	4.51%	5.81%	1.93%	6.05%	6.05%
7	12.34%	3.75%	4.43%	2.40%	7.82%	4.77%
8	7.20%	3.29%	3.68%	0.83%	5.98%	3.68%
9	5.09%	2.86%	2.92%	0.47%	6.07%	2.90%
10	3.56%	2.50%	2.42%	0.99%	4.43%	2.29%
11	2.83%	2.25%	1.98%	0.45%	3.69%	1.73%
12	2.27%	1.99%	1.51%	0.50%	2.54%	1.41%
13	2.05%	1.77%	1.27%	0.78%	1.88%	1.13%
14	1.72%	1.68%	1.01%	0.25%	1.30%	0.77%
15	1.30%	1.40%	0.83%	0.39%	1.02%	0.65%
16	1.16%	1.38%	0.69%	0.32%	0.73%	0.52%
17	0.94%	1.23%	0.58%	0.18%	0.61%	0.44%
18	0.69%	1.10%	0.42%	0.34%	0.44%	0.29%
19	0.55%	1.00%	0.34%	0.23%	0.28%	0.21%
20	0.45%	0.89%	0.31%	0.17%	0.22%	0.18%
21	0.36%	0.81%	0.22%	0.28%	0.17%	0.14%
22	0.28%	0.80%	0.19%	0.13%	0.13%	0.09%
23	0.23%	0.78%	0.16%	0.13%	0.10%	0.08%
24	0.26%	0.63%	0.13%	0.20%	0.08%	0.08%
25	0.13%	0.65%	0.09%	0.10%	0.07%	0.05%
26	0.14%	0.63%	0.10%	0.18%	0.05%	0.05%
27	0.12%	0.57%	0.06%	0.14%	0.03%	0.04%
28	0.08%	0.51%	0.04%	0.10%	0.04%	0.02%
29	0.06%	0.48%	0.05%	0.18%	0.02%	0.02%
30	0.04%	0.40%	0.03%	0.07%	0.03%	0.01%
31	0.04%	0.38%	0.03%	0.08%	0.01%	0.01%
32	0.03%	0.41%	0.02%	0.13%	0.01%	0.02%
33	0.02%	0.36%	0.01%	0.06%	0.01%	0.01%
34	0.03%	0.34%	0.01%	0.06%	0.01%	0.00%
35	0.02%	0.29%	0.02%	0.09%	0.00%	0.01%
36	0.00%	0.27%	0.01%	0.05%	0.00%	0.00%

Note: The Gauss program used to compute these frequencies will be made available upon request.

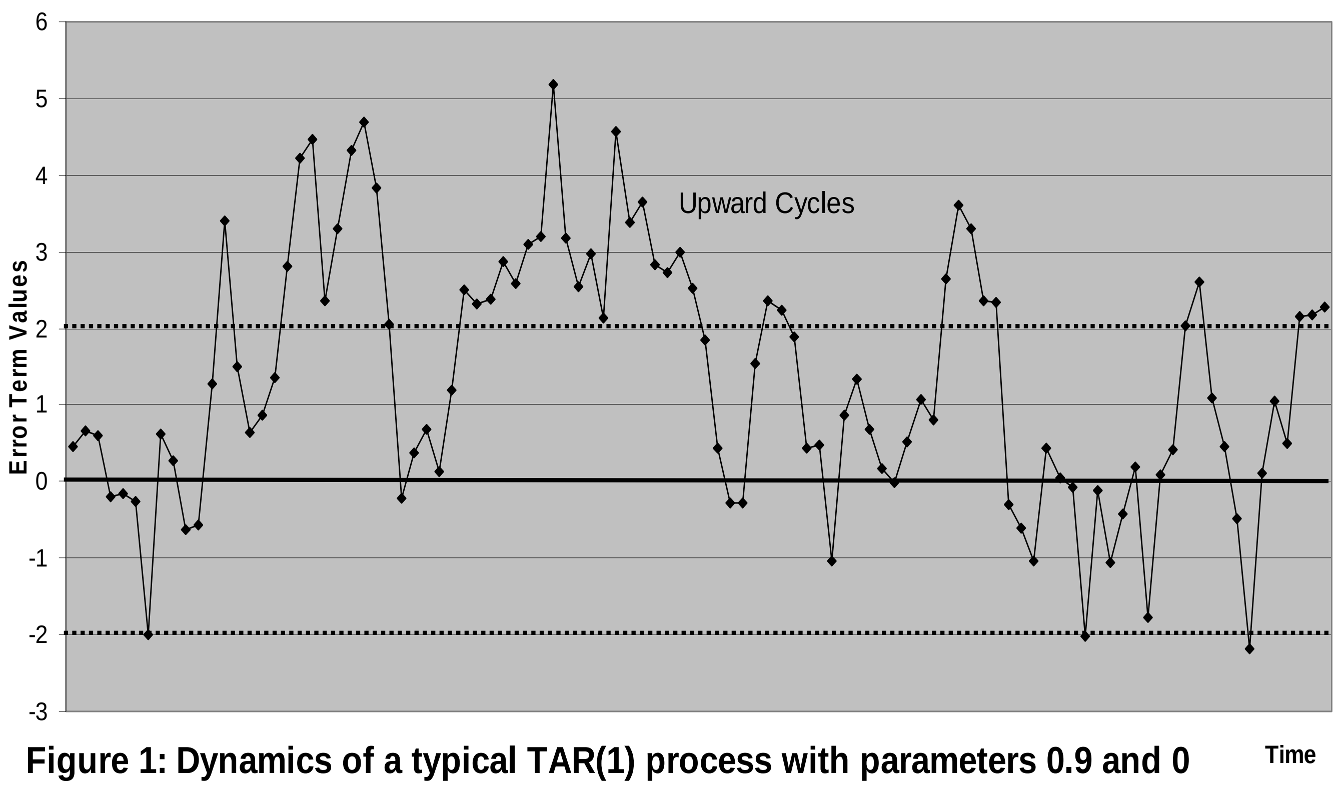

Figure 1: Dynamics of a typical TAR(1) process with parameters 0.9 and 0

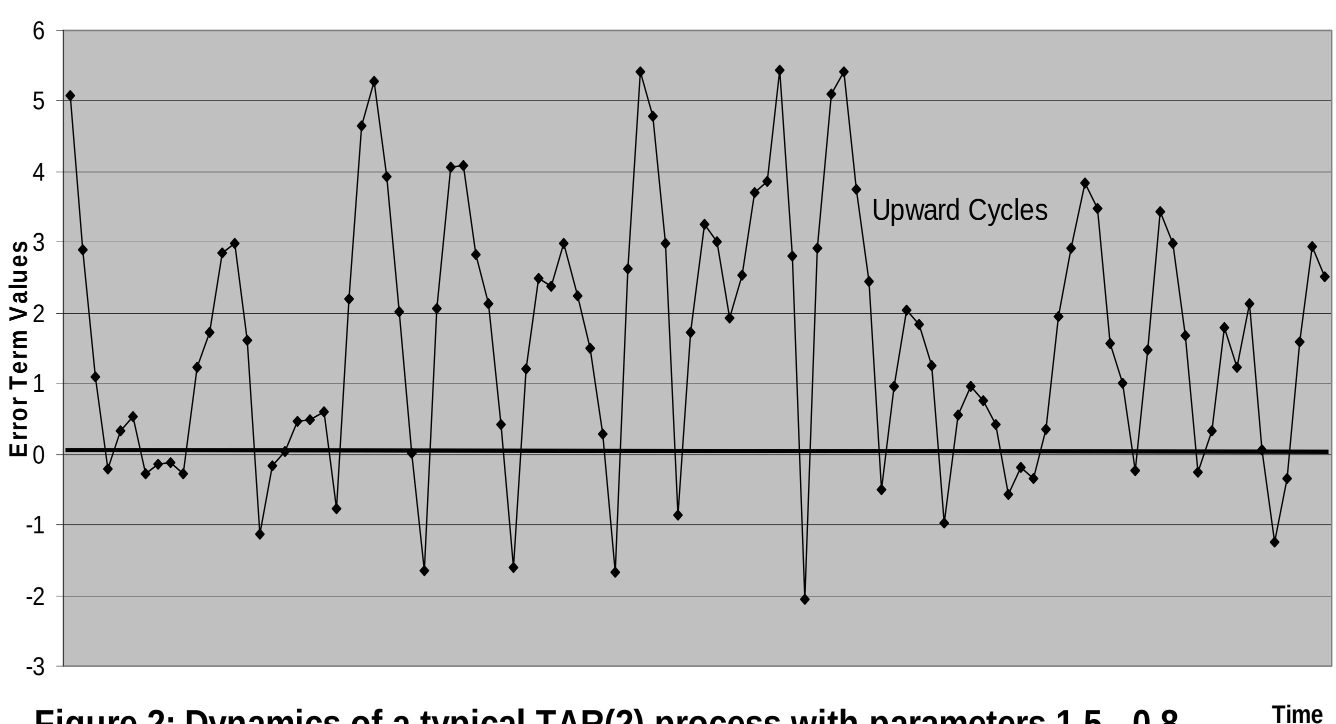

Figure 2: Dynamics of a typical TAR(2) process with parameters 1.5, -0.8, -0.9, and 0.0

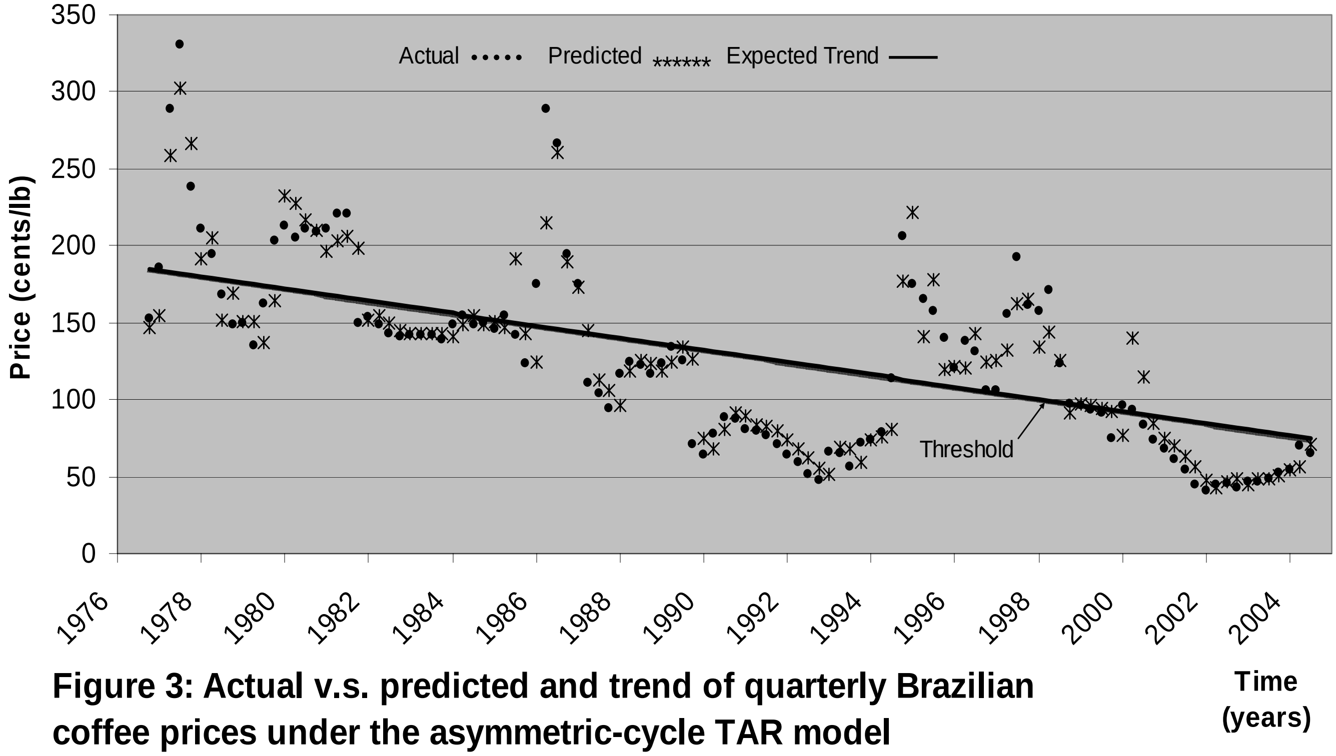

Figure 3: Actual v.s. predicted and trend of quarterly Brazilian coffee prices under the asymmetric-cycle TAR model

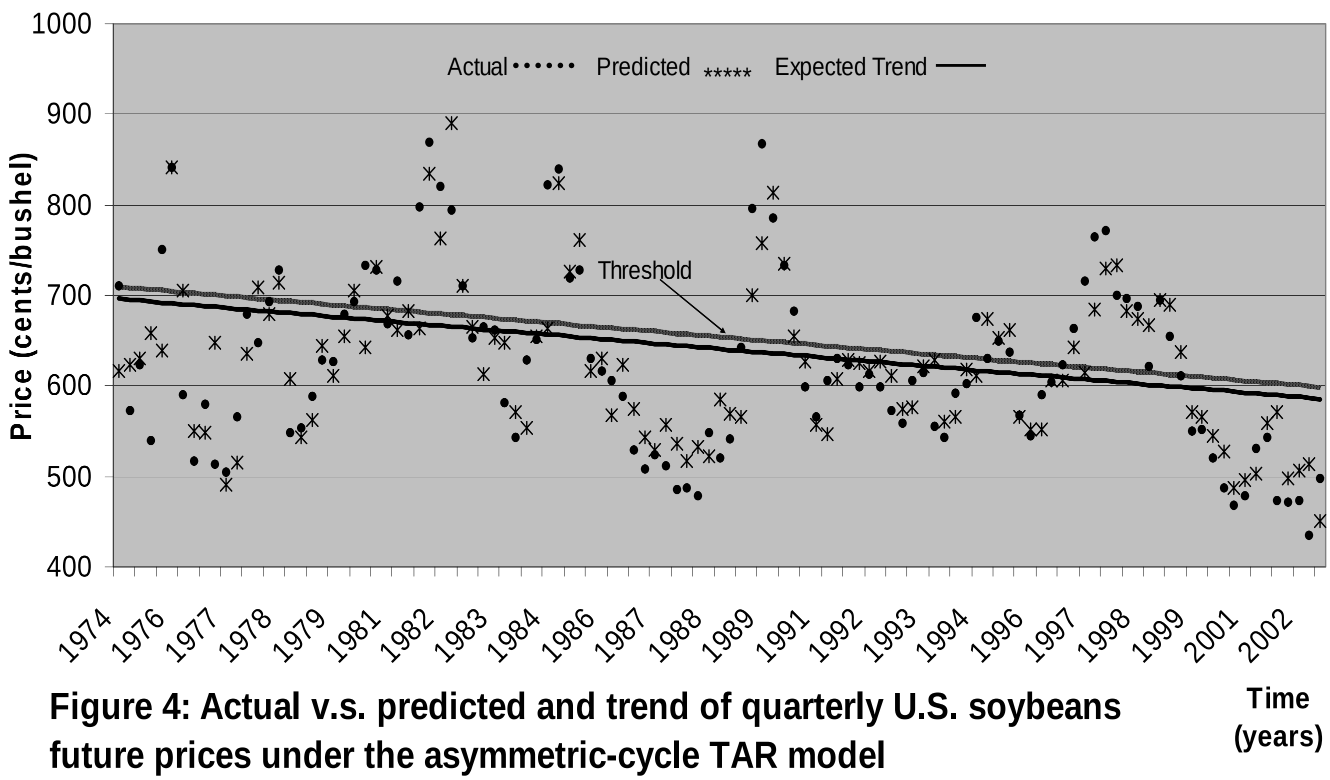

Figure 4: Actual v.s. predicted and trend of quarterly U.S. soybeans future prices under the asymmetric-cycle TAR model